Math 220  Quiz 8 (take-home)  Fall 2019

Name  Solutions

(circle your TA discussion section)

▷ AD1, TR 11:00-12:50, Mina Nahvi
▷ AD2, TR 9:00-10:50, Adriana Morales
▷ AD3, TR 1:00-2:50, Vincent Villalobos
▷ AD4, TR 9:00-9:50, Phuong "Sophie" Le
▷ ADA, TR 8:00-8:50, Scott Harman
▷ ADB, TR 9:00-9:50, Lutian Zhao
▷ ADC, TR 10:00-10:50, Lutian Zhao
▷ ADD, TR 11:00-11:50, Dara Zirlin
▷ ADE, TR 12:00-12:50, David Altizio
▷ ADF, TR 1:00-1:50, Saaber Pourmotabbed
▷ ADG, TR 2:00-2:50, John "Connor" Grady
▷ ADH, TR 3:00-3:50, Sarah Simpson
▷ ADI, TR 4:00-4:50, Ryan McConnell
▷ ADJ, TR 9:00-9:50, Robert "Bob" Krueger
▷ ADK, TR 10:00-10:50, Sarah Simpson
▷ ADL, TR 11:00-11:50, Rocco Davino
▷ ADM, TR 12:00-12:50, Dara Zirlin
▷ ADN, TR 1:00-1:50, John "Connor" Grady
▷ ADO, TR 2:00-2:50, Shuyu "Sonya" Xiao
▷ ADQ, TR 10:00-10:50, Saaber Pourmotabbed
▷ ADR, TR 9:00-9:50, Scott Harman
▷ ADS, TR 12:00-12:50, Rocco Davino
▷ ADT, TR 2:00-2:50, Ryan McConnell
▷ ADU, TR 3:00-3:50, Shuyu "Sonya" Xiao
▷ ADW, TR 8:00-8:50, Robert "Bob" Krueger

- You may lose points if you do not circle your correct discussion section.
- You may work with other MATH 220 students. However each student should write their solutions separately and independently – nobody should copy someone else's work.
- You may use your notes, the textbook, or information found on my course home page including old test and quiz solutions.
- You are not allowed to use a calculator, Wolfram Alpha, or any similar technology.
- There is a higher expectation for the quality of your work on a take-home quiz. Everything should be written logically and legibly with sufficient work to justify each answer. Blank copies of the quiz are available on the course home page.
- Be sure that the pages are nicely stapled – do not just fold the corners.
- The quiz is due at the beginning of your lecture period on Monday, November 4th.
- TAs and Tutors – Do not help students with these specific problems until the quizzes have been collected for all MATH 220 lectures (10-10:50am, 1-1:50pm, 3-3:50pm).
1. (2 points) Determine a formula for \( f(x) \) given that it satisfies the following conditions.

- \( f'(x) = \frac{6x^5 + x^3 - 5x + 8}{x^2 + 1} \) (use polynomial long division)
- \( f(1) = 2\pi - 5 \)

\[
\begin{array}{c}
6x^3 - 5x \\
\hline
6x^5 + x^3 - 5x + 8 \\
- (6x^5 + 6x^3) \\
\hline
-5x^3 - 5x + 8 \\
-(-5x^3 - 5x) \\
\hline
8
\end{array}
\]

\[
f'(x) = \frac{6x^5 + x^3 - 5x + 8}{x^2 + 1} = 6x^3 - 5x + \frac{8}{x^2 + 1}
\]

\[
f(x) = 6\cdot \frac{1}{4}x^4 - 5\cdot \frac{1}{2}x^2 + 8 \cdot \arctan(x) + C
\]

\[
f(x) = \frac{3}{2}x^4 - \frac{5}{2}x^2 + 8\arctan(x) + C
\]

\[
f(1) = 2\pi - 5
\]

\[
\left(\frac{3}{2}\right)(1)^4 - \frac{5}{2}(1)^2 + 8\arctan(1) + C = 2\pi - 5
\]

\[
\frac{3}{2} - \frac{5}{2} + 8 \cdot \frac{\pi}{4} + C = 2\pi - 5 \Rightarrow C = -4
\]

\[
f(x) = \frac{3}{2}x^4 - \frac{5}{2}x^2 + 8\arctan(x) - 4
\]
2. (2 points) Evaluate the following limit. Use proper notation in each step.

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{2k^4 \left( 6 - \frac{n}{k} \right) - (4kn)^2}{k^2 n^3} \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{12k^4 - 2k^3 n - 16k^2 n^2}{k^2 n^3} \\
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{12k^4}{k^2 n^3} - \frac{2k^3 n}{k^2 n^3} - \frac{16k^2 n^2}{k^2 n^3} \right) \\
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{12k^2}{n^3} - \frac{2k}{n^2} - \frac{16}{n} \right) \\
= \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{12k^2}{n^3} - \sum_{k=1}^{n} \frac{2k}{n^2} - \sum_{k=1}^{n} \frac{16}{n} \right) \\
= \lim_{n \to \infty} \left( \frac{12}{n^3} \sum_{k=1}^{n} k^2 - \frac{2}{n^2} \sum_{k=1}^{n} k - \frac{16}{n} \sum_{k=1}^{n} 1 \right) \\
= \lim_{n \to \infty} \left( \frac{12}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{2}{n^2} \frac{n(n+1)}{2} - \frac{16}{n} \right) \\
= \lim_{n \to \infty} \left( \frac{24n^3 + 36n^2 + 12n}{6n^3} - \frac{2n^2 + 2n}{2n^2} - \frac{16n}{n} \right) \\
= \lim_{n \to \infty} \left( \frac{24n^3 + 36n^2 + 12n}{6n^3} - \frac{2n^2 + 2n}{2n^2} - \frac{16n}{n} \right) \\
= \lim_{n \to \infty} \left( 4 + \frac{6}{n} + \frac{2}{n^2} - 1 - \frac{1}{n} - 16 \right) \\
= 4 + 0 + 0 - 1 - 0 - 16 \\
= -13
\]
3. (2 points) From section 5.2 we have the following property of definite integrals.

If \( f(x) \) is continuous and \( m \leq f(x) \leq M \) for \( a \leq x \leq b \), then \( m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M(b-a) \)

Use this property to carefully explain why the following inequality holds.

\[
0.5 \leq \int_{-1}^{8} \left( 15 + \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) \right)^{-1} \, dx \leq 0.75
\]

\( \text{(range of } \arctan \text{)} \quad -\frac{\pi}{2} < \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) < \frac{\pi}{2} \)

\( \text{(mult., by } \frac{6}{\pi} \text{)} \quad -3 < \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) < 3 \)

\( \text{(add 15)} \quad 12 < 15 + \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) < 18 \)

\( \text{(For pos, } a, b \text{)} \quad \frac{1}{12} > \frac{1}{15 + \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right)} > \frac{1}{18} \)

\( \frac{1}{18} < \left( 15 + \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) \right)^{-1} < \frac{1}{12} \)

\[
\int_{-1}^{8} \frac{1}{18} \, dx < \int_{-1}^{8} \left( 15 + \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) \right)^{-1} \, dx < \int_{-1}^{8} \frac{1}{12} \, dx
\]

\[
\frac{1}{18} (8 - (-1)) < \int_{-1}^{8} \left( 15 + \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) \right)^{-1} \, dx < \frac{1}{12} (8 - (-1))
\]

\[0.5 < \int_{-1}^{8} \left( 15 + \frac{6}{\pi} \arctan \left( \frac{3\sqrt{42} + (5x)^99}{9} \right) \right)^{-1} \, dx < 0.75\]

\( \text{Note: For } \int_{-1}^{8} \frac{1}{18} \, dx \text{ and } \int_{-1}^{8} \frac{1}{12} \, dx \text{, you can use FTC or geometry.} \)
4. (2 points) At time \( t \) seconds, the velocity of an object is given by \( v(t) = 50t^3e^{-t} \) meters per second. Using proper notation, express the distance in meters traveled by this object between times \( t = 1 \) and \( t = 5 \) in the following two ways.

(i) A definite integral. Do not evaluate this integral.
\[
distance = \int_{1}^{5} 50t^3e^{-t} \, dt
\]

(ii) A limit of right Riemann sums. Do not evaluate this limit.
\[
distance = \lim_{n \to \infty} \sum_{k=1}^{n} v\left(\frac{k}{n}\right) \Delta t
\]
\[
distance = \lim_{n \to \infty} \sum_{k=1}^{n} 50\left(\frac{k}{n}\right)^3e^{-\left(\frac{k}{n}\right)} \Delta t
\]
\[
distance = \lim_{n \to \infty} \sum_{k=1}^{n} 50\left(1+\frac{k}{n}\right)^3e^{-\left(1+\frac{k}{n}\right)} \frac{4}{n}
\]
5. (2 points) Suppose that \( f(x) \) is continuous at all real numbers and satisfies the following equations.

- \( \int_{-2}^{2} 2f(x) \, dx = 26 \quad \Rightarrow \quad \int_{-2}^{2} f(x) \, dx = \frac{26}{2} = 13 \)
- \( \int_{-2}^{2} 3f(x) \, dx = -15 \quad \Rightarrow \quad \int_{-2}^{2} f(x) \, dx = -\frac{15}{3} = -5 \)
- \( \int_{-2}^{2} 4f(x) \, dx = -4 \quad \Rightarrow \quad \int_{-2}^{2} f(x) \, dx = -\frac{4}{4} = -1 \)

What is the value of \( \int_{-2}^{2} (2 - 5f(x)) \, dx \)?

\[
\int_{-2}^{2} (2 - 5f(x)) \, dx = \int_{-2}^{2} 2 \, dx - 5 \int_{-2}^{2} f(x) \, dx
\]

\[
= 6 - 5 \int_{-2}^{2} f(x) \, dx
\]

\[= 6 - 5(9) = -39 \] (see below)

\[
\int_{-2}^{2} f(x) \, dx = \int_{-2}^{0} f(x) \, dx + \int_{0}^{2} f(x) \, dx
\]

\[
= \int_{-2}^{0} f(x) \, dx + (\int_{0}^{2} f(x) \, dx + \int_{-2}^{2} f(x) \, dx)
\]

\[
= \int_{-2}^{0} f(x) \, dx + (-\int_{0}^{2} f(x) \, dx + \int_{-2}^{2} f(x) \, dx)
\]

\[= 13 + (-(-1)) + (-(-5)) = 9 \]

we used properties \( \int_{b}^{a} f(x) \, dx = -\int_{a}^{b} f(x) \, dx \)

and \( \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \)