Math 220 (section AD?) Quiz 10 Fall 2015

Name Solutions

- You have 20 minutes
- No calculators
- Show sufficient work

1. (2 points) Precisely state *The Mean Value Theorem.*

Suppose f is continuous on $[a,b]$ and differentiable on (a,b).

Then $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ for some c in (a,b)

2. (2 points) Evaluate the definite integral. Simplify your answer.

$$\int_0^{\pi/6} \tan^3(x)\sec^2(x)\,dx \overset{\star}{=} \int_0^{1/\sqrt{3}} u^3\,du$$

$$= \left[\frac{1}{4}u^4\right]_0^{1/\sqrt{3}}$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}\right)^4 - \frac{1}{4}(0)^4$$

$$= \frac{1}{36}$$

Method 1

u-substitution

$u = \tan x$

$du = \sec^2 x\,dx$

at $x=0$, $u=\tan(0) = 0$

at $x=\frac{\pi}{6}$, $u=\tan\left(\frac{\pi}{6}\right) = \dfrac{\sin(\pi/6)}{\cos(\pi/6)} = \dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}}$

Method 2

$$\int_0^{\pi/6} \tan^3 x \sec^2 x\,dx = \int_0^{\pi/6} \tan^2 x \sec x \sec x \tan x\,dx$$

u-substitution

$u = \sec x$

$du = \sec x \tan x\,dx$

at $x=0$, $u = \sec(0) = 1$

at $x = \frac{\pi}{6}$, $u = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$

$$= \int_0^{\pi/6} (\sec^2 x - 1)\sec x \sec x \tan x\,dx$$

$$= \int_1^{2/\sqrt{3}} (u^2-1)\,u\,du$$

$$= \int_1^{2/\sqrt{3}} (u^3-u)\,du$$

$$= \left[\frac{1}{4}u^4 - \frac{1}{2}u^2\right]_1^{2/\sqrt{3}}$$

$$= \left[\frac{1}{4}\left(\frac{2}{\sqrt{3}}\right)^4 - \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2\right] - \left[\frac{1}{4} - \frac{1}{2}\right]$$

$$= \left[\frac{1}{4}\cdot\frac{16}{9} - \frac{1}{2}\cdot\frac{4}{3}\right] - \left[-\frac{1}{4}\right]$$

$$= \frac{1}{36}$$

Method 3

$$\int_0^{\pi/6} \tan^3 x \sec^2 x\,dx = \int_0^{\pi/6} \frac{\sin^3 x}{\cos^3 x}\cdot\frac{1}{\cos^2 x}\,dx$$

u-substitution

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

at $x=0$, $u = \cos(0) = 1$

at $x = \frac{\pi}{6}$, $u = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$$= \int_0^{\pi/6} \frac{\sin^2 x}{\cos^5 x}\sin x\,dx$$

$$= \int_0^{\pi/6} \frac{1-\cos^2 x}{\cos^5 x}\sin x\,dx$$

$$= \int_1^{\sqrt{3}/2} \frac{1-u^2}{u^5}(-du)$$

$$= \int_1^{\sqrt{3}/2} (-u^{-5} + u^{-3})\,du$$

$$= \left[\frac{1}{4}u^{-4} - \frac{1}{2}u^{-2}\right]_1^{\sqrt{3}/2} = \cdots = \frac{1}{36}$$

3. (2 points each) Evaluate the indefinite integrals.

(a) $\int \frac{6x^5}{x^3+1}\,dx$ $= \int \frac{2x^3}{x^3+1} \cdot 3x^2\,dx$

Method 1

u-substitution

$u = x^3+1 \Rightarrow x^3 = u-1$

$du = 3x^2\,dx$

$= \int \frac{2(u-1)}{u}\,du$

$= \int \left(2 - 2\cdot\frac{1}{u}\right) du$

$= 2u - 2\ln|u| + C$

$= 2(x^3+1) - 2\ln|x^3+1| + C$

Method 2 (polynomial long division)

$$\require{enclose} \begin{array}{r} 6x^2 \\ x^3+1 \enclose{longdiv}{6x^5} \\ \underline{6x^5+6x^2} \\ -6x^2 \end{array}$$

$\int \frac{6x^5}{x^3+1}\,dx = \int \left(6x^2 - \frac{6x^2}{x^3+1}\right) dx = \int 6x^2\,dx - \int \frac{6x^2}{x^3+1}\,dx$

$= 2x^3 - 2\ln|x^3+1| + C$

using $\int \frac{6x^2}{x^3+1}\,dx = \int \frac{2}{u}\,du$ (u-sub $u = x^3+1$)

(b) $\int \frac{\sqrt{x}+e^{\sqrt{x}}}{\sqrt{x}}\,dx$ $= \int \left(\sqrt{x}+e^{\sqrt{x}}\right)\cdot\frac{1}{\sqrt{x}}\,dx$

u-substitution

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= \int (u+e^u)\cdot 2\,du$

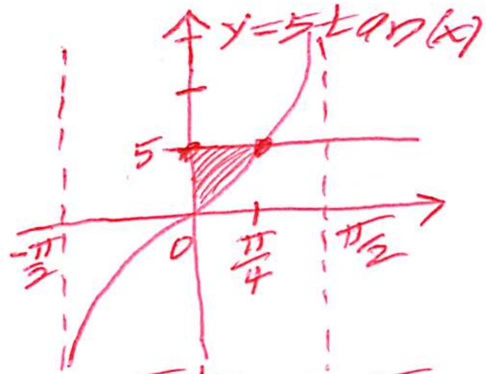
$= \int (2u+2e^u)\,du$

$= u^2 + 2e^u + C$

$= (\sqrt{x})^2 + 2e^{\sqrt{x}} + C$

$= x + 2e^{\sqrt{x}} + C$

4. (2 points) Let **R** be the finite region bounded by $y = 5\tan(x)$, $y = 5$, and $x = 0$. In the following manner, set up but do not evaluate definite integrals which represent the area of the region **R**.



$5\tan x = 5$

$\tan x = 1$

$x = \pi/4$

(a) Integrate with respect to x.

$$\text{area} = \int_a^b (y_{top} - y_{bottom})\,dx$$

$$\text{area} = \int_0^{\pi/4} (5 - 5\tan(x))\,dx$$

(b) Integrate with respect to y. (The integrands in parts (a) and (b) should be different.)

$$y = 5\tan(x) \Rightarrow x = \arctan(y/5)$$

$$\text{area} = \int_c^d (x_{right} - x_{left})\,dy$$

$$\text{area} = \int_0^5 \left(\arctan\left(\tfrac{y}{5}\right) - 0\right)dy$$