You may work with other MATH 220 students. However each student should write up solutions separately and independently – nobody should copy someone else’s work.

You may use your notes, the textbook, or information found on my course home page.

You may use a calculator only for basic arithmetic. In particular you should not use its graphing features.

You are not allowed to search the Internet, use Wolfram Alpha, or use technology for anything beyond what is stated above.

There is a higher expectation for the quality of your work on a take-home quiz. Everything should be written logically and legibly with sufficient work to justify each answer. Blank copies of the quiz are available on the course home page.

Be sure that the pages are nicely stapled – do not just fold the corners.

The quiz is due at the beginning of your official lecture period on Friday, October 17.

Note to TAs and Tutors – you should not help students with these specific problems or go over solutions until after 5pm Friday.
1. (2 points) Evaluate \( \lim_{x \to \infty} \left( 1 + \frac{2}{3x^2} \right)^{6x^2} \) (Indeterminate Form 1^\infty)

\[
\lim_{x \to \infty} \left( 1 + \frac{2}{3x^2} \right)^{6x^2} = \lim_{x \to \infty} e^{\ln \left( \left( 1 + \frac{2}{3x^2} \right)^{6x^2} \right)}
\]

\[
= e^{\lim_{x \to \infty} \ln \left( 1 + \frac{2}{3x^2} \right)^{6x^2}}
\]

\[
= e^{\lim_{x \to \infty} (6x^2 \ln \left( 1 + \frac{2}{3x^2} \right))}
\]

\[
= e^{\lim_{x \to \infty} \frac{6\ln \left( 1 + \frac{2}{3x^2} \right)}{x^{-2}}}
\]

\[
= e^{\lim_{x \to \infty} \frac{\frac{d}{dx} \left[ 6\ln \left( 1 + \frac{2}{3x^2} \right) \right]}{\frac{d}{dx} \left( x^{-2} \right)}}
\]

\[
= e^{\lim_{x \to \infty} \left( \frac{6 \cdot \frac{1}{1 + \frac{2}{3x^2}} \cdot -\frac{2}{3} x^{-3}}{-2 x^{-3}} \right)}
\]

\[
= e^{\lim_{x \to \infty} \left( \frac{4}{1 + \frac{2}{3x^2}} \right)}
\]

\[
= e^4
\]
2. (3 points) For each \( x > 0 \), let \( m(x) \) be the slope of the line which goes through the point \((0, 0)\) and the point \((x, y)\) on the curve \( y = x^2 e^{-0.25x} \).

What is the largest possible value for \( m(x) \)?

\[
\begin{align*}
\text{slope } &= \frac{\Delta y}{\Delta x} \\
&= \frac{y - 0}{x - 0} \\
&= \frac{x^2 e^{-0.25x} - 0}{x - 0} \\
&= xe^{-0.25x}
\end{align*}
\]

we must \( \text{maximize } m(x) = xe^{-0.25x} \) for \( x > 0 \)

\[
\begin{align*}
m'(x) &= \frac{d}{dx}(x) \cdot e^{-0.25x} + x \cdot \frac{d}{dx}(e^{-0.25x}) \\
&= 1 \cdot e^{-0.25x} + x \cdot -0.25 e^{-0.25x} \\
&= e^{-0.25x} (1 - 0.25x)
\end{align*}
\]

always positive equals 0 when \( x = 4 \)

The only critical number is \( x = 4 \)

<table>
<thead>
<tr>
<th>Values of ( m'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>++ + 0 --- -- - + +</td>
</tr>
</tbody>
</table>

\( x \)

The maximum value of \( m(x) \) occurs for \( x = 4 \)

\[
\begin{align*}
\text{largest slope of such a line } &= m(4) = 4e^{-0.25(4)} \\
&= 4e^{-1} \\
&= \frac{4}{e}
\end{align*}
\]
3. (3 points) What are the coordinates \((x, y)\) for the highest point on the graph of the function 
\[ f(x) = \frac{e^{6x}}{e^{ax} + 4} \] ?

Maximize \( f(x) = \frac{e^{6x}}{e^{ax} + 4} \) for \(-\infty < x < \infty\)

\[
f'(x) = \frac{6e^{6x}(e^{ax} + 4) - e^{6x}(ae^{ax})}{(e^{ax} + 4)^2}
\]

\[
= \frac{3e^{6x}[2(e^{ax} + 4) - 3e^{ax}]}{(e^{ax} + 4)^2}
\]

\[
= \frac{3e^{6x}[8 - e^{ax}]}{(e^{ax} + 4)^2}
\]

Since \(3e^{6x}\) and \((e^{ax} + 4)^2\) are positive for all \(x\)

The only critical number is when \(8 - e^{ax} = 0\) That is \(e^{ax} = 8 \Rightarrow ax = \ln 8 \Rightarrow x = \frac{\ln 8}{a} = \frac{12}{3} = \frac{4}{3}\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f'(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty)</td>
<td>++</td>
</tr>
<tr>
<td>(\frac{12}{3})</td>
<td>0</td>
</tr>
<tr>
<td>(+\infty)</td>
<td>--</td>
</tr>
</tbody>
</table>

Values of \(f'(x)\)

\[ f\left(\frac{12}{3}\right) = \left(\frac{12}{3}, \frac{e^{6 \cdot \frac{12}{3}}}{e^{\frac{a \cdot 12}{3}} + 4}\right) = \left(\frac{12}{3}, \frac{4}{12}\right) = \left(\frac{12}{3}, \frac{1}{3}\right) \]

Highest point is \(\left(\frac{12}{3}, \frac{1}{3}\right)\)

\[ f\left(\frac{12}{3}\right) = \left(\frac{12}{3}, \frac{e^{6 \cdot \frac{12}{3}}}{e^{\frac{a \cdot 12}{3}} + 4}\right) = \left(\frac{12}{3}, \frac{4}{12}\right) = \left(\frac{12}{3}, \frac{1}{3}\right) \]
4. (2 points) Complete the sentences concerning the function \( f(x) = 3 + 4xe^{-5x} \).

(a) The function \( f \) is decreasing on the interval \( [\frac{1}{5}, \infty) \)
(b) The function \( f \) is increasing on the interval \( (-\infty, \frac{1}{5}) \)
(c) The function \( f \) is concave down on the interval \( (-\infty, \frac{2}{5}) \)
(d) The function \( f \) is concave up on the interval \( (\frac{2}{5}, \infty) \)

\[
f'(x) = 4e^{-5x} + 4xe^{-5x}
= 4e^{-5x}(1 - 5x)
\]

\[
f''(x) = 4e^{-5x}(-5)(1 - 5x) + 4e^{-5x}
= -20e^{-5x}(1 - 5x + 1)
= -20e^{-5x}(2 - 5x)
\]

values of \( f'(x) \)

<table>
<thead>
<tr>
<th>( + + + )</th>
<th>( 0 )</th>
<th>( - - - )</th>
</tr>
</thead>
<tbody>
<tr>
<td>incr. ( \frac{1}{5} )</td>
<td>( F )</td>
<td>decr.</td>
</tr>
</tbody>
</table>

values of \( f''(x) \)

<table>
<thead>
<tr>
<th>( -- - )</th>
<th>( 0 )</th>
<th>( + + + )</th>
</tr>
</thead>
<tbody>
<tr>
<td>conc. down ( \frac{2}{5} )</td>
<td>( F )</td>
<td>conc. up</td>
</tr>
</tbody>
</table>