SOLUTIONS

- Sit in your assigned seat (circled below).
- Circle your TA discussion section.
- Do not open this test booklet until I say START.
- Turn off all electronic devices and put away all items except a pen/pencil and an eraser.
- Remove hats and sunglasses.
- You must show sufficient work to justify each answer.
- While the test is in progress, we will not answer questions concerning the test material.
- Do not leave early unless you are at the end of a row.
- Quit working and close this test booklet when I say STOP.
- Quickly turn in your test to me or a TA and show your Student ID.

> AD1, TR 11:00-12:50, Hannah Kolb Spinoza
> AD3, TR 1:00-2:50, Michael Santana
> ADB, TR 9:00-9:50, Ziyi Pan
> ADD, TR 11:00-11:50, Lisa Hickok
> ADF, TR 1:00-1:50, Jiao Liang
> ADH, TR 3:00-3:50, Lechao Xiao
> ADJ, TR 9:00-9:50, Meghan Galiardi
> ADL, TR 11:00-11:50, Andrew McConvey
> ADN, TR 1:00-1:50, Benjimen Fulan
> ADP, TR 3:00-3:50, Hongfei Tian
> ADR, TR 9:00-9:50, Noah Chartoff
> ADT, TR 2:00-2:50, Anna Weigandt

> AD2, TR 9:00-10:50, Ki Yeun Kim
> ADA, TR 8:00-8:50, Ziyi Pan
> ADC, TR 10:00-10:50, Lisa Hickok
> ADE, TR 12:00-12:50, Andrew McConvey
> ADG, TR 2:00-2:50, Derrek Yager
> ADI, TR 4:00-4:50, Lechao Xiao
> ADK, TR 10:00-10:50, Meghan Galiardi
> ADM, TR 12:00-12:50, Benjamin Fulan
> ADO, TR 2:00-2:50, Jiao Liang
> ADQ, TR 4:00-4:50, Hongfei Tian
> ADS, TR 12:00-12:50, Derrek Yager
> ADU, TR 3:00-3:50, Anna Weigandt

FRONT OF ROOM – 228 Natural History Building
1. (5 points) If the point \((5, -3)\) is on the graph of a one-to-one function \(f\), then which one of the following points must be on the graph of \(y = f^{-1}(x)\) ?

(a) \((3, 5)\)

(b) \((3, -5)\)

\[\text{f is one-to-one so it has an inverse } f^{-1}\]

\[f(5) = -3 \Rightarrow f^{-1}(-3) = 5\]

\[\text{Thus } (-3, 5) \text{ is on the graph of } f^{-1}\]

(c) \((-3, 5)\)

(d) \((-3, -5)\)

(e) \((5, 3)\)

(f) \((-5, 3)\)

(g) \((-5, -3)\)

2. (5 points) Which one of the following statements is true?

(a) A function which is continuous at a point \(a\) must be continuous at all other points in the domain of the function.

\[\text{False}\]

(b) A function which is continuous at all points in its domain must be one-to-one.

\[\text{False}\]

(c) A function which is one-to-one must be increasing on its domain.

\[\text{False}\]

(d) A function which is continuous at a point \(a\) must also be differentiable at \(a\).

\[\text{False}\]

(e) A function which is differentiable at a point \(a\) must also be continuous at \(a\).

\[\text{True - see Theorem on page 158 in book}\]

(f) A function which is differentiable at a point \(a\) must be differentiable at all other points in the domain of the function.

\[\text{False}\]
3. (10 points) Let \( f(x) = 6x - 4x^2 \).

Use the definition of a derivative as a limit to prove that \( f'(x) = 6 - 8x \).

Show each step in your calculation and be sure to use proper terminology in each step of your proof.

\[
f'(x) = \lim_{{h \to 0}} \frac{{f(x+h) - f(x)}}{h}
\]

\[
= \lim_{{h \to 0}} \frac{{6(x+h) - 4(x+h)^2 - (6x - 4x^2)}}{h}
\]

\[
= \lim_{{h \to 0}} \frac{{6x + 6h - 4(x^2 + 2xh + h^2) - 6x + 4x^2}}{h}
\]

\[
= \lim_{{h \to 0}} \frac{{6x + 6h - 4x^2 - 8xh - 4h^2 - 6x + 4x^2}}{h}
\]

\[
= \lim_{{h \to 0}} \frac{{6h - 8xh - 4h^2}}{h}
\]

\[
= \lim_{{h \to 0}} \frac{{h(6 - 8x - 4h)}}{h}
\]

\[
= \lim_{{h \to 0}} (6 - 8x - 4h)
\]

\[
= 6 - 8x
\]
4. (10 points) Determine a formula for an exponential function which goes through the point \((2, 3)\) and has a y-intercept of 8.

\[ y = C \cdot a^x \]

y-intercept is 8 \(\Rightarrow (0, 8)\) is on graph of \(y\).

\((x, y) = (2, 3) \Rightarrow 3 = 8 \cdot a^2 \Rightarrow a^2 = \frac{3}{8} \Rightarrow a = \left(\frac{3}{8}\right)^{1/2}\)

\[ y = 8 \cdot \left(\frac{3}{8}\right)^{x/2} \Rightarrow y = 8 \cdot \left(\frac{3}{8}\right)^{x/2} \]

5. (10 points) What is the value of \(\sin(\tan^{-1}(4))\)?

Let \(\theta = \tan^{-1}(4)\)

so \(\tan \theta = 4\) with \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\) (actually \(0 < \theta < \frac{\pi}{2}\) since \(\tan \theta > 0\))

From Pythagorean Theorem

\[ \sin(\tan^{-1}(4)) = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17} \]

6. (10 points) Given that \(0 < \theta < \frac{\pi}{2}\), and \(\sin \theta = \frac{1}{3}\), evaluate \(\cos(\pi - \theta)\).

\[ \cos(\pi - \theta) = -\cos \theta \]

\[ = -\sqrt{\cos^2 \theta} \]

\[ = -\sqrt{1 - \sin^2 \theta} \]

\[ = -\sqrt{1 - (\frac{1}{3})^2} \]

\[ = -\sqrt{\frac{8}{9}} \]

\[ = -\frac{2\sqrt{2}}{3} \]
7. (10 points) Determine a formula for \( f^{-1}(x) \) given that \( f(x) = \frac{1 - e^x}{2 + 3e^x} \)

\[
\begin{align*}
  y &= \frac{1 - e^x}{2 + 3e^x} \\
  x &= \frac{1 - e^y}{2 + 3e^y} \\
  x(2 + 3e^y) &= 1 - e^y \\
  2x + 3xe^y &= 1 - e^y \\
  3xe^y + ye^y &= 1 - 2x \\
  (3x + 1)e^y &= 1 - 2x
\end{align*}
\]

\[ f^{-1}(x) = \ln \left( \frac{1 - 2x}{3x + 1} \right) \]

\[ y = \ln \left( \frac{1 - 2x}{3x + 1} \right) \]

8. (10 points) Solve for \( x \) in the equation below.

\[ e^{2 + \ln(x+2)} = 15e^{2 - \ln x} \]

\[
\begin{align*}
  \ln(e^{2 + \ln(x+2)}) &= \ln(15e^{2 - \ln x}) \\
  2 + \ln(x+2) &= \ln(15) + 2 - \ln(x) \\
  \ln(x) + \ln(x+2) &= \ln(15) \\
  \ln(x(x+2)) &= \ln(15) \\
  x(x+2) &= 15 \\
  x^2 + 2x - 15 &= 0 \\
  (x-3)(x+5) &= 0
\end{align*}
\]

\[ x = 3 \text{ or } x = -5 \]

but \( x = -5 \) is not in domain, so \( x = 3 \) is only solution.

9. (10 points) Find all horizontal asymptotes on the graph of \( f(x) = \frac{8 - 3e^x}{4e^x + 2} \)

\[
\begin{align*}
  \lim_{x \to \infty} \frac{8 - 3e^x}{4e^x + 2} &= \lim_{x \to \infty} \frac{(8 - 3e^x) \cdot \frac{1}{e^x}}{(4e^x + 2) \cdot \frac{1}{e^x}} \\
    &= \lim_{x \to \infty} \frac{8e^{-x} - 3}{4 + 7e^{-x}} \\
    &= \frac{8}{4} = 2
\end{align*}
\]

Thus \( y = \frac{3}{4} \) and \( y = 4 \) are horizontal asymptotes.
10. (5 points each) Evaluate the following limits without the use of derivatives. Show sufficient justification for each answer. An answer of ‘does not exist’ is not sufficient. For infinite limits you must state if it is $\infty$ or $-\infty$.

(a) \[ \lim_{x \to 0} \frac{6 + 2e^x}{7 - 5\cos x} = \frac{6 + 2}{7 - 5} = \frac{8}{2} = 4 \]

(b) \[ \lim_{x \to \infty} \frac{8 \sin x}{3x} \]

-1 $\leq \sin x \leq 1$

-8 $\leq 8\sin x \leq 8$

$-\frac{8}{3x} \leq \frac{8\sin x}{3x} \leq \frac{8}{3x}$ for $x > 0$

\[ \lim_{x \to \infty} \left(-\frac{8}{3x}\right) = 0 \quad \text{and} \quad \lim_{x \to \infty} \left(\frac{8}{3x}\right) = 0 \]

By the Squeeze Theorem,

\[ \lim_{x \to \infty} \frac{8\sin x}{3x} = 0 \]
(c) \[ \lim_{x \to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4} \to 0 \]
= \lim_{x \to 2^-} \frac{x \cdot (x-2)}{(x-2)^2}
= \lim_{x \to 2^-} \frac{x}{x-2} \to 0^-
= -\infty

(d) \[ \lim_{x \to 2^-} \frac{\frac{1}{2} - \frac{1}{x}}{x^2 - 4} \]
= \lim_{x \to 2^-} \frac{\frac{x}{2x} - \frac{2}{2x}}{x^2 - 4}
= \lim_{x \to 2^-} \frac{x-2}{2x \cdot (x-2)(x+2)}
= \lim_{x \to 2^-} \frac{x-2}{2x(x-2)(x+2)}
= \lim_{x \to 2^-} \frac{1}{2x(x+2)}
= \frac{1}{2 \cdot 2(2+2)} \text{ since } \frac{1}{2x(x+2)} \text{ is continuous at 2}
= \frac{1}{16}
Students – do not write on this page!

1. (5 points) ______________________________

2. (5 points) ______________________________

3. (10 points) ______________________________

4. (10 points) ______________________________

5. (10 points) ______________________________

6. (10 points) ______________________________

7. (10 points) ______________________________

8. (10 points) ______________________________

9. (10 points) ______________________________

10a. (5 points) ______________________________

10b. (5 points) ______________________________

10c. (5 points) ______________________________

10d. (5 points) ______________________________

**TOTAL (100 points) ________________**