Test 1 solutions

1. (12 points) Given that $f(x) = 2x^3 + 5$, find a formula for $f^{-1}(x)$.

$$y = f^{-1}(x)$$

$$f(y) = x$$

$$2y^3 + 5 = x$$

$$2y^3 = x - 5$$

$$y^3 = \frac{x-5}{2}$$

$$y = \sqrt[3]{\frac{x-5}{2}}$$

$$f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$$

2. (12 points) Suppose $f(x) = 2 - \ln x$ and $g(x) = \sqrt{x}$. Determine a formula and find the domain for $(g \circ f)(x)$.

$$(g \circ f)(x) = g(f(x))$$

$$= g(2 - \ln x)$$

$$= \sqrt{2 - \ln x}$$

From $\ln x$, we see that $x > 0$

From $\sqrt{2 - \ln x}$, we see that $2 - \ln x \geq 0$

$$\ln x \leq 2$$

$$x \leq e^2$$

The domain of $(g \circ f)(x)$ is $(0, e^2]$

3. (9 points) Which one of the following equations must hold in order for a function f to be continuous at a number a ?

(a) $\lim_{x\to 0} f(x) = a$
(b) $\lim_{x\to 0} f(x) = 0$
(c) $\lim_{x\to 0} f(x) = f(a)$
(d) $\lim_{x\to 0} f(x) = f'(a)$
(e) $\lim_{x\to a} f(x) = a$
(f) $\lim_{x\to a} f(x) = 0$
(g) $\lim_{x\to a} f(x) = f(a)$
(h) $\lim_{x\to a} f(x) = f'(a)$
(i) $\lim_{x\to \infty} f(x) = a$
(j) $\lim_{x\to \infty} f(x) = 0$
(k) $\lim_{x\to \infty} f(x) = f(a)$
(l) $\lim_{x\to \infty} f(x) = f'(a)$

4. (6 points) Given a function $f(x)$ for which $\lim_{h\to 0} \dfrac{f(-5+h)-f(-5)}{h}$ exists, which one of the following statements must be true?

(a) f is continuous but not differentiable at $x = -5$.
(b) f is differentiable but not continuous at $x = -5$.
(c) f is both differentiable and continuous $x = -5$.
(d) f is neither continuous nor differentiable at $x = -5$.
(e) f is continuous but not differentiable at $x = 0$.
(f) f is differentiable but not continuous at $x = 0$.
(g) f is both differentiable and continuous $x = 0$.
(h) f is neither continuous nor differentiable at $x = 0$.
(i) f is continuous but not differentiable at $x = 5$.
(j) f is differentiable but not continuous at $x = 5$.
(k) f is both differentiable and continuous $x = 5$.
(l) f is neither continuous nor differentiable at $x = 5$.

$f'(-5) = \lim_{h\to 0} \dfrac{f(-5+h)-f(-5)}{h}$

since the limit exists, f is differentiable at -5.

Differentiability implies continuity

5. (12 points) Let $f(x) = x^3 - 5x$. Use the definition of a derivative as a limit to show that $f'(x) = 3x^2 - 5$. Show each step in your calculation and be sure to use proper terminology.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - 5(x+h) - (x^3 - 5x)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h - x^3 + 5x}{h}$$

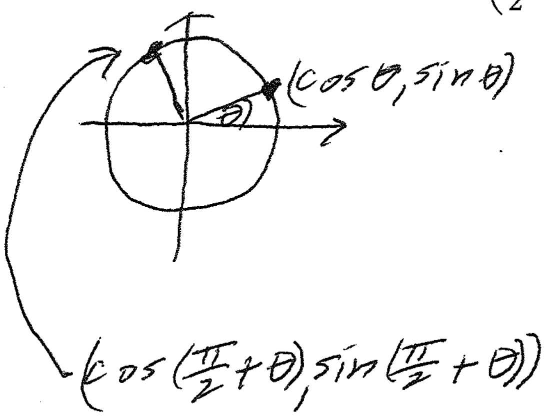
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 5h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 5)}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 5)$$

$$= 3x^2 - 5$$

6. (6 points) For a given angle θ, it is known that $\cos\theta \approx 0.927$, $\sin\theta \approx 0.375$ and $\tan\theta \approx 0.404$. What is the value of $\cos\left(\frac{\pi}{2}+\theta\right)$?



From unit circle, we see

$\cos(\frac{\pi}{2}+\theta) = -\sin\theta$

More formally,

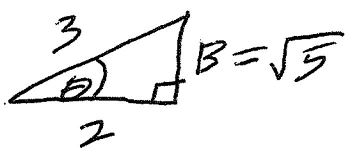
$$\cos(\tfrac{\pi}{2}+\theta) = \cos(\tfrac{\pi}{2})\cos\theta - \sin(\tfrac{\pi}{2})\sin\theta$$
$$= 0\cdot\cos\theta - 1\cdot\sin\theta$$
$$= -\sin\theta$$

Thus $\cos(\frac{\pi}{2}+\theta) = -0.375$

7. (6 points) Evaluate and simplify $\tan\left(\cos^{-1}(2/3)\right)$.

$\theta = \cos^{-1}(\frac{2}{3})$ $\quad \theta \in [0, \pi]$

$\cos(\theta) = \frac{2}{3}$ $\quad \theta \in [0, \frac{\pi}{2}]$



$2^2 + B^2 = 3^2$

so $B = \sqrt{5}$

$\tan(\cos^{-1}(\frac{2}{3})) = \tan\theta$

$= \frac{\sqrt{5}}{2}$ $\left(\frac{\text{opp}}{\text{adj}}\right)$

8. (7 points) Determine real numbers a and b so that the expression $8\csc^2\theta - 5\cot^2\theta$ can be rewritten as $a\csc^2\theta + b$.

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$

$\Rightarrow 1 + \cot^2\theta = \csc^2\theta$

Thus $8\csc^2\theta - 5\cot^2\theta =$

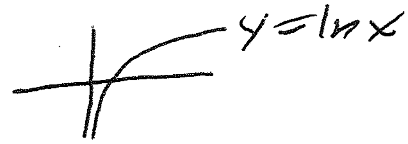
$8\csc^2\theta - 5(\csc^2\theta - 1) =$

$3\csc^2\theta + 5$

so $a = 3$, $b = 5$

9. (5 points each) Evaluate the following limits and simplify each answer. An answer of 'does not exist' is not sufficient. If the limit is infinite then you must state if it is ∞ or $-\infty$.

(a) $\displaystyle\lim_{x\to 0} \frac{2}{e^x+3}$ $\displaystyle= \frac{2}{e^0+3} = \frac{2}{1+3} = \frac{2}{4} = \frac{1}{2}$



(b) $\displaystyle\lim_{x\to 2^+} (1000 + 5\ln(x-2))$

$x \to 2^+ \Rightarrow x-2 \to 0^+ \Rightarrow \ln(x-2) \to -\infty$

Thus $\displaystyle\lim_{x\to 2^+} (1000+5\ln(x-2)) = -\infty$

(c) $\displaystyle\lim_{x\to 3/2} \frac{4x^2-9}{2x-3}$ $\displaystyle= \lim_{x\to 3/2} \frac{(2x+3)(2x-3)}{2x-3}$

$\displaystyle= \lim_{x\to 3/2} (2x+3)$

$= 2\left(\frac{3}{2}\right) + 3$

$= 6$

(d) $\lim\limits_{x\to 2^-} \dfrac{5-3x}{x-2}$ $\to -1$

$\to 0^-$

$= \infty$

(e) $\lim\limits_{x\to\infty} \dfrac{(2x+1)^2}{(3x+1)^2} = \lim\limits_{x\to\infty} \dfrac{4x^2+4x+1}{9x^2+6x+1}$

$= \lim\limits_{x\to\infty} \dfrac{(4x^2+4x+1)\left(\frac{1}{x^2}\right)}{(9x^2+6x+1)\left(\frac{1}{x^2}\right)}$

$= \lim\limits_{x\to\infty} \dfrac{4+\frac{4}{x}+\frac{1}{x^2}}{9+\frac{6}{x}+\frac{1}{x^2}} = \dfrac{4}{9}$

(f) $\lim\limits_{x\to 5} \left(\dfrac{1}{x-5} - \dfrac{10}{x^2-25}\right)$

$= \lim\limits_{x\to 5} \left(\dfrac{x+5}{(x-5)(x+5)} - \dfrac{10}{x^2-25}\right)$

$= \lim\limits_{x\to 5} \dfrac{\cancel{x-5}}{\cancel{(x-5)}(x+5)}$

$= \lim\limits_{x\to 5} \dfrac{1}{x+5}$

$= \dfrac{1}{10}$