1. (10 points) Let \( f(x) = 4x^2 - 9 \). Use the definition of a derivative as a limit to show that \( f'(x) = 8x \). Show each step in your calculation and be sure to use proper terminology.

\[
\begin{align*}
\frac{f'(x)}{1} & = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
& = \lim_{h \to 0} \frac{(4(x+h)^2 - 9) - (4x^2 - 9)}{h} \\
& = \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 9 - 4x^2 + 9}{h} \\
& = \lim_{h \to 0} \frac{8xh + 4h^2}{h} \\
& = \lim_{h \to 0} (8x + 4h) \\
& = 8x
\end{align*}
\]

2. (10 points) Evaluate the following derivatives.

(a) \( \frac{d}{dx} (\sec x) = \sec x \tan x \)

(b) \( \frac{d}{dx} (\tan x) = \sec^2 x \)

(c) \( \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2} \)

(d) \( \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \)

(e) \( \frac{d}{dx} (e^x) = e^x \)
3. (6 points) Find \( g'(t) \) given that \( g(t) = 12t^3 - 8t^2 + 13t - 32 \)

\[
g'(t) = 36t^2 - 16t + 13
\]

4. (6 points) Find \( f'(x) \) given that \( f(x) = x^6 \ln x \)

\[
f'(x) = \frac{(x^6)'(\ln x) + (x^6)(\ln x)'}{(x^6)' + (\ln x)'}
\]

\[
f'(x) = 6x^5 \ln x + x^6 \cdot \frac{1}{x}
\]

\[
f'(x) = 6x^5 \ln x + x^5
\]

5. (6 points) Find \( P'(t) \) given that \( P(t) = 30(t^6 - 5t^3 + 8)^4 \)

\[
P'(t) = 120(t^6 - 5t^3 + 8)^3(6t^5 - 15t^2)
\]
6. (4 points) Find \( \frac{dy}{dx} \) given that \( y = x^{4x} \)

\[
\ln y = \ln(x^{4x}) \\
\ln y = 4x \ln x \\
\frac{1}{y} \cdot \frac{dy}{dx} = 4\ln x + 4x \cdot \frac{1}{x} \\
\frac{dy}{dx} = y \left(4\ln x + 4\right) \\
\frac{dy}{dx} = x^{4x} \left(4\ln x + 4\right)
\]

7. (4 points) Find \( \frac{dy}{dx} \) given that \( \sin y = \frac{x}{y} \)

\[
\frac{d}{dx}(\sin y) = \frac{d}{dx}(\frac{x}{y}) \\
\cos y \cdot \frac{dy}{dx} = \frac{\frac{d}{dx}(x) \cdot y - \frac{d}{dx}(y) \cdot x}{y^2} \\
\cos y \cdot \frac{dy}{dx} = \frac{y - \frac{dy}{dx} \cdot x}{y^2} \\
\frac{y^2 \cos y}{dx} = y - \frac{dy}{dx} \cdot x \\
(y^2 \cos y + x) \cdot \frac{dy}{dx} = y \\
\frac{dy}{dx} = \frac{y}{y^2 \cos y + x}
\]
8. (4 points) The graph of a function \( y = f(x) \) has the property that the slope of the tangent line at each point on this graph is equal to twice its \( y \)-coordinate. If the graph goes through the point \((0, 5)\), then find a formula for \( f(x) \).

\[
\frac{dy}{dx} = 2y \quad \text{HAS SOLUTION}
\]

\[y = Ce^{2x}\]

Plugging in \((x, y) = (0, 5)\) gives

\[5 = Ce^{2 \cdot 0} \Rightarrow C = 5\]

Thus, \( y = 5e^{2x} \)

9. (10 points) A spherical balloon is inflated at a rate of 150 cubic feet per minute. How quickly is the balloon’s radius increasing at the instant the radius is 5 feet?

\[V = \frac{4}{3} \pi r^3\]

\[\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}\]

\[150 = 4 \pi (5)^2 \cdot \frac{dr}{dt}\]

\[\frac{dr}{dt} = \frac{150}{100 \pi} = \frac{3}{2 \pi} \text{ ft/min}\]
10. (5 points) The graph of $f(x)$ is shown below.

Circle the graph of $f'(x)$, given that it is one of the six choices below.
11. (8 points) Find each value of $x$ on the interval $[0, 2\pi]$ at which the graph of $y = \sin x + \cos^2 x$ has a horizontal tangent line.

\[ y' = \cos x + 2\cos x(-\sin x) \]
\[ 0 = \cos x (1 - 2\sin x) \]
\[ \cos x = 0 \text{ or } 1 - 2\sin x = 0 \]
\[ \sin x = \frac{1}{\sqrt{2}} \]
\[ x = \frac{\pi}{4}, \frac{3\pi}{4} \]

\[ x = \frac{\pi}{6}, \frac{5\pi}{6} \]

12. (7 points) What are the coordinates $(x, y)$ for the highest point on the graph of the function $g(x) = \frac{\ln x}{x}$. Be sure each coordinate is in simplified form.

\[ g'(x) = \frac{(\ln x)'(x) - (\ln x)(x)'}{x^2} \]
\[ g'(x) = \frac{(x)(\ln x)' - (\ln x)(x)}{x^2} \]
\[ g'(x) = \frac{1 - \ln x}{x^2} \]

VALUES OF $g'$

<table>
<thead>
<tr>
<th>$+$</th>
<th>$0$</th>
<th>$-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{e}$</td>
<td>$\frac{1 - \ln x}{x^2}$</td>
<td></td>
</tr>
</tbody>
</table>

$0 = \frac{1 - \ln x}{x^2} \Rightarrow 1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$

$(x, y) = (e, \frac{\ln e}{e}) = \left(e, \frac{1}{e}\right)$
13. (5 points) A function $f(x)$ is given below along with its first and second derivatives in factored and unfactored forms.

- $f(x) = x^4 - 4x^3 + 16x - 16 = (x + 2)(x - 2)^3$
- $f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2$
- $f''(x) = 12x^2 - 24x = 12x(x - 2)$

The graph of $f(x)$ is concave down upon which one of the following intervals?

(a) $(-2, 2)$
(b) $(-1, 2)$
(c) $(0, 2)$
(d) $(-\infty, -2)$
(e) $(-\infty, -1)$
(f) $(-\infty, 0)$
(g) $(-2, \infty)$
(h) $(-1, \infty)$
(i) $(0, \infty)$
(j) $(-\infty, \infty)$

14. (5 points) A function $g(x)$ has the following derivative.

$$g'(x) = 5e^x(x - 1)^2(x - 2)^3(x - 3)^4$$

Which one of the following statements is true about the graph of $g(x)$?

(a) There is a local minimum at $x = -1$
(b) There is a local minimum at $x = 0$
(c) There is a local minimum at $x = 1$
(d) There is a local minimum at $x = 2$
(e) There is a local minimum at $x = 3$
(f) There is a local maximum at $x = -1$
(g) There is a local maximum at $x = 0$
(h) There is a local maximum at $x = 1$
(i) There is a local maximum at $x = 2$
(j) There is a local maximum at $x = 3$
15. (5 points) From a height of 8 feet, a ball is thrown straight up with an initial velocity of 16 feet per second. Until it hits the ground, the ball’s height in feet above ground level is given by \( h = -16t^2 + 16t + 8 \) where \( t \) is the number of seconds after the ball is thrown. What is the maximum height above ground level attained by the ball?

(a) 5 feet
(b) 6 feet
(c) 8 feet
(d) 9 feet
(e) 12 feet
(f) 16 feet
(g) 24 feet
(h) 32 feet

\[
\begin{align*}
h' &= -32t + 16 \\
0 &= -32t + 16 \\
32t &= 16 \\
t &= \frac{1}{2}
\end{align*}
\]

VALUES OF \( h' \)

\[
\begin{array}{c|c}
\text{t} & \text{h'} \\
\hline
1/8 & 0 \\
1/2 & -4 \\
1 & -8 \\
\end{array}
\]

16. (5 points) If \( f \) is an even function which is differentiable everywhere, then show very clearly how Rolle’s Theorem can be used to prove the existence of a real number \( c \) for which \( f'(c) = 0 \).

\[ f \text{ is differentiable everywhere} \]
\[ \implies f \text{ is continuous everywhere} \]
\[ \text{Since } f \text{ is even, we have that } f(-a) = f(a) \text{ for all } a \text{ in the domain of } f. \]

In particular, we have \( f(-1) = f(1) \)

we have:

1. \( f \) is continuous on \([-1, 1]\)
2. \( f \) is differentiable on \((-1, 1)\)
3. \( f(-1) = f(1) \)

By Rolle’s Theorem, there is a \( c \) in \((-1, 1)\) such that \( f'(c) = 0 \)