MATH 220 (sections AL1, BL1, DL1, L1, L2, L3, L4) **Test 2** **Fall 2009**

1. (10 points) Let $f(x) = 4x^2-9$. Use the definition of a derivative as a limit to show that $f'(x) = 8x$. Show each step in your calculation and be sure to use proper terminology.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(4(x+h)^2-9)-(4x^2-9)}{h}$$

$$= \lim_{h\to 0} \frac{4x^2+8xh+4h^2-9-4x^2+9}{h}$$

$$= \lim_{h\to 0} \frac{h(8x+4h)}{h}$$

$$= \lim_{h\to 0} (8x+4h)$$

$$= 8x$$

2. (10 points) Evaluate the following derivatives.

(a) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(b) $\frac{d}{dx}(\tan x) = \sec^2 x$

(c) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

(e) $\frac{d}{dx}(e^x) = e^x$

3. (6 points) Find $g'(t)$ given that $g(t) = 12t^3 - 8t^2 + 13t - 32$

$$\boxed{g'(t) = 36t^2 - 16t + 13}$$

4. (6 points) Find $f'(x)$ given that $f(x) = x^6 \ln x$

$$f'(x) = \underline{(x^6)'(\ln x) + (x^6)(\ln x)'}$$

$$f'(x) = 6x^5 \ln x + x^6 \cdot \cancel{\frac{1}{x}}$$

$$\boxed{f'(x) = 6x^5 \ln x + x^5}$$

5. (6 points) Find $P'(t)$ given that $P(t) = 30\left(t^6 - 5t^3 + 8\right)^4$

$$\boxed{P'(t) = 120(t^6 - 5t^3 + 8)^3 \cdot (6t^5 - 15t^2)}$$

6. (4 points) Find $\frac{dy}{dx}$ given that $y = x^{4x}$

$$\ln y = \ln(x^{4x})$$
$$\ln y = 4x \ln x$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = 4\ln x + 4x\cdot\frac{1}{x}$$
$$\frac{dy}{dx} = y\,(4\ln x + 4)$$
$$\boxed{\frac{dy}{dx} = x^{4x}(4\ln x + 4)}$$

7. (4 points) Find $\frac{dy}{dx}$ given that $\sin y = \frac{x}{y}$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{x}{y}\right)$$
$$\cos y\cdot\frac{dy}{dx} = \frac{\frac{d}{dx}(x)\cdot y - \frac{d}{dx}(y)\cdot x}{y^2}$$
$$\cos y\cdot\frac{dy}{dx} = \frac{y - \frac{dy}{dx}\cdot x}{y^2}$$
$$y^2\cos y\,\frac{dy}{dx} = y - \frac{dy}{dx}\cdot x$$
$$(y^2\cos y + x)\cdot\frac{dy}{dx} = y$$
$$\boxed{\frac{dy}{dx} = \frac{y}{y^2\cos y + x}}$$

8. (4 points) The graph of a function $y = f(x)$ has the property that the slope of the tangent line at each point on this graph is equal to twice its y-coordinate. If the graph goes through the point $(0, 5)$, then find a formula for $f(x)$.

$\frac{dy}{dx} = 2y$ HAS SOLUTION

$y = Ce^{2x}$

PLUGGING IN $(x, y) = (0, 5)$ GIVES

$5 = Ce^{2 \cdot 0} \Rightarrow C = 5$

THUS, $\boxed{y = 5e^{2x}}$

9. (10 points) A spherical balloon is inflated at a rate of 150 cubic feet per minute. How quickly is the balloon's radius increasing at the instant the radius is 5 feet?

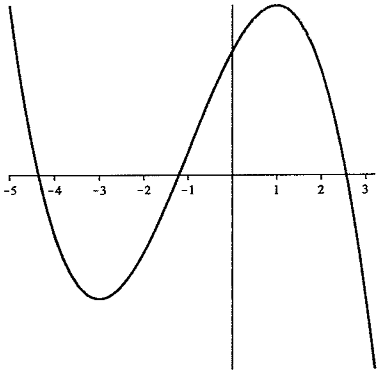
$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

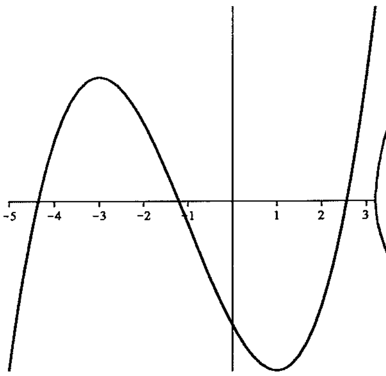
$150 = 4\pi(5)^2 \cdot \frac{dr}{dt}$

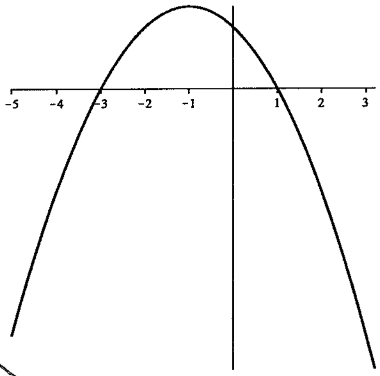
$\frac{dr}{dt} = \frac{150}{100\pi} = \boxed{\frac{3}{2\pi} \text{ ft/min}}$

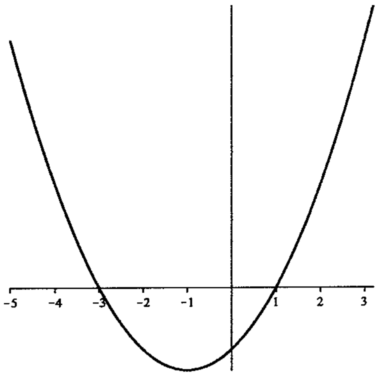
10. (5 points) The graph of $f(x)$ is shown below.

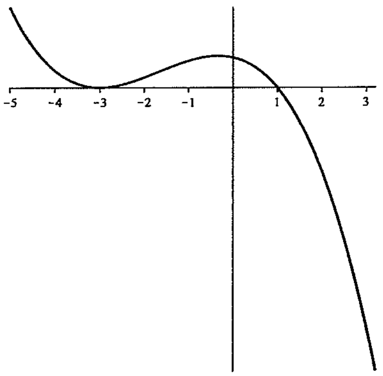


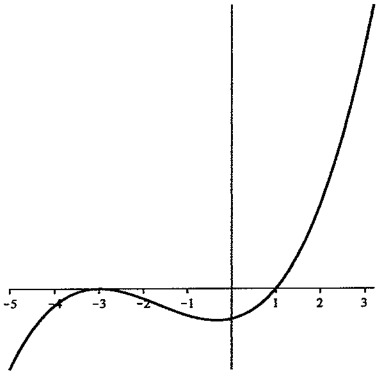
Circle the graph of $f'(x)$, given that it is one of the six choices below.

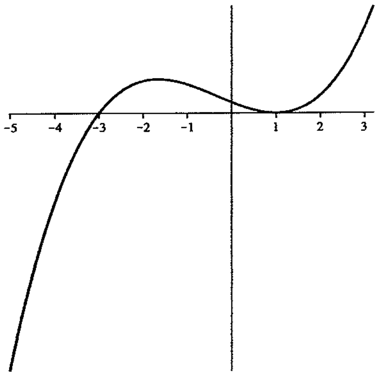












11. (8 points) Find each value of x on the interval $[0, 2\pi]$ at which the graph of $y = \sin x + \cos^2 x$ has a horizontal tangent line.

$$y' = \cos x + 2\cos x(-\sin x)$$

$$0 = \cos x(1 - 2\sin x)$$

$$\cos x = 0 \quad \text{OR} \quad 1 - 2\sin x = 0$$

$$\sin x = 1/2$$

$$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}} \qquad \boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$$

12. (7 points) What are the coordinates (x, y) for the highest point on the graph of the function $g(x) = \dfrac{\ln x}{x}$. Be sure each coordinate is in simplified form.

$$g'(x) = \frac{(\ln x)'(x) - (\ln x)(x)'}{x^2}$$

$$g'(x) = \frac{(\frac{1}{x})(x) - (\ln x)(1)}{x^2}$$

$$g'(x) = \frac{1 - \ln x}{x^2}$$

VALUES OF g'

$+ + \quad 0 \quad - -$

$e \qquad x$

$$0 = \frac{1 - \ln x}{x^2} \Rightarrow 1 - \ln x = 0$$

$$\Rightarrow \ln x = 1 \Rightarrow x = e$$

$$(x, y) = \left(e, \frac{\ln e}{e}\right) = \boxed{\left(e, \frac{1}{e}\right)}$$

13. (5 points) A function $f(x)$ is given below along with its first and second derivatives in factored and unfactored forms.

- $f(x) = x^4 - 4x^3 + 16x - 16 = (x+2)(x-2)^3$
- $f'(x) = 4x^3 - 12x^2 + 16 = 4(x+1)(x-2)^2$
- $f''(x) = 12x^2 - 24x = 12x(x-2)$

VALUES OF f''

$+ + 0 - - - - 0 + +$

$0 \quad 2 \quad x$

The graph of $f(x)$ is concave down upon which one of the following intervals?

(a) $(-2, 2)$

(b) $(-1, 2)$

[✓ (c) $(0, 2)$]

(d) $(-\infty, -2)$

(e) $(-\infty, -1)$

(f) $(-\infty, 0)$

(g) $(-2, \infty)$

(h) $(-1, \infty)$

(i) $(0, \infty)$

(j) $(-\infty, \infty)$

14. (5 points) A function $g(x)$ has the following derivative.

$$g'(x) = 5e^x(x-1)^2(x-2)^3(x-3)^4$$

Which one of the following statements is true about the graph of $g(x)$?

VALUES OF g'

$- - 0 - - 0 + + 0 + + +$

$1 \quad 2 \quad 3$

(a) There is a local minimum at $x = -1$

(b) There is a local minimum at $x = 0$

(c) There is a local minimum at $x = 1$

[✓ (d) There is a local minimum at $x = 2$]

(e) There is a local minimum at $x = 3$

(f) There is a local maximum at $x = -1$

(g) There is a local maximum at $x = 0$

(h) There is a local maximum at $x = 1$

(i) There is a local maximum at $x = 2$

(j) There is a local maximum at $x = 3$

15. (5 points) From a height of 8 feet, a ball is thrown straight up with an initial velocity of 16 feet per second. Until it hits the ground, the ball's height in feet above ground level is given by $h = -16t^2 + 16t + 8$ where t is the number of seconds after the ball is thrown. What is the maximum height above ground level attained by the ball?

- (a) 5 feet
- (b) 6 feet
- (c) 8 feet
- (d) 9 feet
- [✓ (e) 12 feet]
- (f) 16 feet
- (g) 24 feet
- (h) 32 feet

$h' = -32t+16$

$0 = -32t+16$

$32t = 16$

$t = \frac{1}{2}$

VALUES OF h'

$++\ 0\ --$

$\xrightarrow[1/2]{} t$

$h(\frac{1}{2}) = -16(\frac{1}{2})^2 + 16(\frac{1}{2}) + 8$

$= -4 + 8 + 8$

$= \boxed{12}$

16. (5 points) If f is an even function which is differentiable everywhere, then show very clearly how Rolle's Theorem can be used to prove the existence of a real number c for which $f'(c) = 0$.

f is differentiable everywhere implies f is continuous everywhere. Since f is even, we have that $f(-a) = f(a)$ for all a in the domain of f.

In particular, we have $f(-1) = f(1)$

we have: (1) f is continuous on $[-1,1]$

(2) f is differentiable on $(-1,1)$

③ $f(-1) = f(1)$

By Rolle's Theorem, there is a c in $(-1,1)$ such that $f'(c) = 0$