1. (10 points) Evaluate the following integral.

\[ \int \frac{dx}{x^2 \sqrt{x^2 + 4}} = \int \frac{2 \sec^2 \theta}{4 \tan \theta \sqrt{4 \tan^2 \theta + 4}} \]

Let \( x = 2 \tan \theta \)
\( dx = 2 \sec^2 \theta \, d\theta \)
\[ \sqrt{x^2 + 4} = \frac{x}{\sqrt{x^2 + 4}} \]
\[ \sin \theta = \frac{x}{\sqrt{x^2 + 4}} \]

\[ = \int \frac{\sec^2 \theta}{\tan \theta} \, d\theta = \int \frac{\sec \theta}{\sin \theta} \, d\theta \]
\[ = \int \sec \theta \, d\theta \]
\[ = \int \frac{1}{\sin \theta} \, d\theta \]
\[ = \int \frac{-1}{4u} + C \]
\[ = \int \frac{-1}{4 \sin \theta} + C \]
\[ = \frac{-\sqrt{x^2 + 4}}{4x} + C \]

2. (10 points) Evaluate the following integral.

\[ \int \frac{dx}{x^2 - 6x + 8} = \int \frac{dx}{(x-4)(x-2)} \]

\[ \frac{1}{x^2 - 6x + 8} = \frac{A}{x-4} + \frac{B}{x-2} \]
\[ 1 = A(x-2) + B(x-4) \]
\[ x=2 \quad 1 = -2B \]
\[ B = -\frac{1}{2} \]
\[ x=4 \quad 1 = 2A \]
\[ A = \frac{1}{2} \]

\[ = \int \left( \frac{\frac{1}{2}}{x-2} + \frac{-\frac{1}{2}}{x-4} \right) \, dx \]
\[ = \frac{1}{2} \ln |x-4| - \frac{1}{2} \ln |x-2| + C \]
3. (10 points) Evaluate the following integral.

\[
\int \frac{dx}{x^2 - 6x + 10} = \int \frac{dx}{(x-3)^2 + 1} = \int \frac{du}{u^2 + 1} \quad \text{WHERE} \quad u = x - 3
\]
\[
= \tan^{-1} u + C = \tan^{-1} (x-3) + C
\]

4. (10 points) Evaluate the following integral.

\[
\int \frac{2x - 5}{x - 3} \, dx = \int \left(2 + \frac{1}{x-3} \right) \, dx = \left(2x + \ln|x-3| \right) + C
\]
5. (10 points) Evaluate the following integral.

\[
\int \frac{2x^2}{(x-1)(x^2+1)} \, dx = \int \left( \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \right) \, dx
\]

\[
= \ln |x-1| + \frac{1}{2} \ln (x^2+1) + \tan^{-1} x + C
\]

6. (10 points) Evaluate the following integral. Use proper notation for each step in your work.

\[
\int_3^\infty \frac{6}{x^3} \, dx = \lim_{b \to \infty} \int_3^b \frac{6}{x^3} \, dx
\]

\[
= \lim_{b \to \infty} \left[ -\frac{1}{x^2} \right]_3^b
\]

\[
= \lim_{b \to \infty} \left[ -\frac{1}{b^2} - \left(-\frac{1}{9}\right) \right]
\]

\[
= 2
\]
7. (10 points) Evaluate the following integral. Use proper notation for each step in your work.

\[
\int_1^3 \frac{dx}{x-1} = \lim_{b \to 1^+} \int_b^3 \frac{dx}{x-1} = \lim_{b \to 1^+} \left[ \ln|x-1| \right]^3_b = \lim_{b \to 1^+} \left[ \ln 2 - \ln b - 1 \right] = +\infty
\]

8. (10 points) Write out the first five terms of the sequence, determine if the sequence converges, and if so find its limit.

\[\left\{ \frac{\ln k}{k} \right\}_{k=2}^{+\infty}\]

\[\frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4}, \frac{\ln 5}{5}, \frac{\ln 6}{6}, \ldots \]

This sequence converges to 0 since \( \lim_{k \to \infty} \frac{\ln k}{k} = 0 \).
9. (10 points) Show that the given sequence is eventually strictly increasing or eventually strictly decreasing. Which one? You must fully justify your claim and include the value that $n$ needs to exceed for this to occur.

$$\{15n - n^2\}_{n=1}^{+\infty}$$

$$f(x) = 15x - x^2$$
$$f'(x) = 15 - 2x$$
$$f'(x) < 0 \text{ when } x > 7.5$$

Thus the sequence $\{15n - n^2\}_{n=1}^{\infty}$ is strictly decreasing for $n \geq 8$.

10. (5 points) Show how to define the sequence below more concisely using curly braces and a general term as in the problem above.

$$\frac{1}{4},\frac{1}{9},\frac{1}{16},\frac{1}{25},\frac{1}{36},\frac{1}{49},\frac{1}{64},\ldots$$

$$\left\{\frac{(-1)^n}{n^2}\right\}_{n=2}^{+\infty}$$

Also $$\left\{\frac{(-1)^{n+1}}{(n+1)^2}\right\}_{n=1}^{+\infty}$$
11. (5 points) Consider the sequence $\{a_n\}_{n=1}^{\infty}$ where

$$
a_1 = \sqrt{2}\\
a_2 = \sqrt{2 + \sqrt{2}}\\
a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}\\
a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}\\
a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}\\$$

\[\vdots\]

(a) Find a recursion formula for $a_{n+1}$.

$$a_1 = \sqrt{2}$$

$$a_{n+1} = \sqrt{2 + a_n} \quad \text{for } n \geq 1$$

(b) Assuming the sequence converges, find its limit.

Suppose $\lim_{n \to \infty} a_n = L$.

Then $\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2 + a_n}$

$$L = \sqrt{2 + L} \Rightarrow (L-2)(L+1) = 0$$

$L^2 - L - 2 = 0$ \[\Rightarrow L = 2 \quad \text{or} \quad L = -1\]

$L = 2$