1. (5 points) A function \( f(x) \) satisfies all of the following conditions:

- \( f(-2) = 256, \ f(-1) = 162, \) and \( f(2) = 0. \)
- \( f'(x) = 5(x + 2)(x - 2)^3. \)
- \( f''(x) = 20(x + 1)(x - 2)^2. \)

(a) Find all critical points for \( f(x) \). Write each answer as an ordered pair \((x, y)\).

We have critical points for \( f(x) \) at \( x = -2 \) and \( x = 2 \), the only \( x \)-values for which \( f'(x) = 0 \). The ordered pairs are \((-2, 256)\) and \((2, 0)\).

(b) Find all inflection points for \( f(x) \). Write each answer as an ordered pair \((x, y)\).

We note that \( f''(x) = 0 \) at \( x = -1 \) and \( x = 2 \) but these are not necessarily inflection points. The concavity must switch at these \( x \)-values in order for \( f(x) \) to have inflection points. By plugging in any \( x \)-value less than \(-1\), we find that \( f''(x) \) is negative. By plugging in any \( x \)-value between \(-1\) and \( 2 \), we find that \( f''(x) \) is positive. By plugging in any \( x \)-value greater than \( 2 \), we find that \( f''(x) \) is positive. Thus the concavity only switched at \( x = -1 \) where the graph of \( f(x) \) switched from concave down to concave up. Thus the only inflection point for \( f(x) \) is the point \((-1, 162)\).

(c) For \(-3 \leq x \leq 3\), sketch a plausible graph of \( f(x) \). Be sure to clearly indicate on your graph the points found in parts \((a)\) and \((b)\).
2. (5 points) A company finds that it can sell 4000 cartons of ice cream when it charges $4.00 per carton. For every $0.25 decrease in price, demand increases by 200 cartons.

(a) Find a formula which relates the number of cartons sold, \( q \), to the price per carton, \( p \).

We first make a table of values for \( p \) and \( q \).

\[
\begin{array}{c|c}
 p & q \\
\hline
4 & 4000 \\
3.75 & 4200 \\
3.5 & 4400 \\
3.25 & 4600 \\
\end{array}
\]

Since \( q \) is a linear function of \( p \), we use the formula \( q - q_0 = m(p - p_0) \). The slope \( m = \frac{4000 - 4200}{4 - 3.75} = -800 \). Thus \( q - 4000 = -800(p - 4) \). We can simplify this to \( q = -800p + 7200 \).

(b) Find a formula for revenue as a function of one variable (either \( q \) or \( p \)).

Revenue = price \cdot quantity = p \cdot (-800p + 7200).

(c) Find the price per carton and the corresponding number of cartons sold which will maximize revenue for this company.

Here is a graph of the revenue function found above.

Using the built-in features of the graphing calculator, we find that we obtain a maximum revenue when \( p = 4.5 \). You can obtain this by hand if you take the derivative of the revenue function and set it equal to 0. Now from our formula in part (a) we get that \( q = -800(4.5) + 7200 = 3600 \).

(d) What is the maximum revenue for this company?

Maximum revenue = \( p \cdot q = (4.5) \cdot (3600) = $16200 \)