Math 415  
Lecture 7  
Sec 2.1  Vector Spaces  
Read Sections 2.1 & 2.2 (in advance if possible)

Much of the structure of linear algebra is designed to study linear problems. In this section we introduce the objects of study: vectors. These are general objects that live in vector spaces. Let us use $V$ to denote a vector space and use $v, w, \ldots$ to denote vectors.

In Chap 7 we introduce linear functions from one vector space to another. Let $X$ and $V$ be two vector spaces and $L : X \rightarrow V$ a linear function. Then we will be interested in linear eqns of the type

\[(*) \quad L(x) = v, \quad x \in X, \; v \in V.\]

Compare this to $Ax = b$. But $*$ could be linear eqns, linear differential equations, linear integral eqns, etc, and we want to intro structures that will apply to them all.

For a vector space we need a set $V$ and two operations, addition denoted $\oplus$ and scalar multiplication denoted $\circ$. Here are some examples:
Example 1: Spaces of n-tuples $\mathbb{R}^n$

$V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $W = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

Define $V \oplus W = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$, $\mathbb{C}V = \begin{pmatrix} c \cdot v_1 \\ \vdots \\ c \cdot v_n \end{pmatrix}$

Example 2: Spaces of Matrices $M_{m \times n}$

$M = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{m1} & \cdots & m_{mn} \end{pmatrix}$, $N = \begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & \ddots & \vdots \\ n_{m1} & \cdots & n_{mn} \end{pmatrix}$

Define $M \oplus N = \begin{pmatrix} m_{11} + n_{11} & \cdots & m_{1n} + n_{1n} \\ \vdots & \ddots & \vdots \\ m_{m1} + n_{m1} & \cdots & m_{mn} + n_{mn} \end{pmatrix}$

$c \cdot M = \begin{pmatrix} c \cdot m_{11} & \cdots & c \cdot m_{1n} \\ \vdots & \ddots & \vdots \\ c \cdot m_{m1} & \cdots & c \cdot m_{mn} \end{pmatrix}$

Example 3: Spaces of Polynomials $P_m^n$

$P_m^n = \{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \}$, degree $n$

If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and $q(x) = b_m x^m + \cdots + b_1 x + b_0$

Define $p(x) \oplus q(x) = (a_n + b_m) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0)$

$c \cdot p(x) = (c \cdot a_n) x^n + \cdots + (c \cdot a_1) x + (c \cdot a_0)$

Adding reals: scalar multi of reals
Example 4. Function Spaces \( \mathcal{F}(S) \) (real fun defined on \( S \))

\[ C(S) = \{ f(x) \mid f(x) \text{ is continuous real valued for all } x \in \mathbb{R}^2 \} \]

Define

\[(f + g)(x) = f(x) + g(x) \quad \forall x \]

\[(c \cdot f)(x) = c \cdot f(x) \quad \forall x. \]

We can talk about all these examples simultaneously in terms of a **Vector Space** \( V \).

A set equipped with two operations satisfying (show transparency and talk about it!)

**Question:** Is the set of upper triangular matrices a vector space?

**Example:** Let \( V \) be a vector space, \( S \) a set of real numbers, and define \( \mathcal{F}(S) = \{ f : S \to V \} \)

Is \( \mathcal{F}(S) \) a vector space? We need operations!

\[(f + g)(x) = f(x) + g(x) \quad \forall x \in S \]

\[+ \text{ in } \mathcal{F} \]

\[(c \cdot f)(x) = c \cdot f(x) \]

\[c \text{ in } V \]

Think about this in terms of functions from \( x \in \mathbb{R} \) to \( 3 \times 3 \) upper triangular.

\[ f(x) = \begin{pmatrix} x & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & x+1 \end{pmatrix} \]

is an element of \( \mathcal{F}(\mathbb{R}, 3 \times 3 \text{ upper triangular}) \).
Definition 2.1

A vector space is a set $V$ equipped with two operations:

(i) Addition: adding any pair of vectors $v, w \in V$ gives another vector $v + w \in V$;

(ii) Scalar Multiplication: multiplying a vector $v \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $cv \in V$;

subject to the following axioms, for all $u, v, w \in V$ and all scalars $c, d \in \mathbb{R}$:

(a) Commutativity of Addition: $v + w = w + v$.

(b) Associativity of Addition: $u + (v + w) = (u + v) + w$.

(c) Additive Identity: There is a zero element $0 \in V$ satisfying $v + 0 = v = 0 + v$.

(d) Additive Inverse: For each $v \in V$ there is an element $-v \in V$ such that $v + (-v) = 0 = (-v) + v$.

(e) Distributivity: $(c + d)v = (cv) + (dv)$, and $c(v + w) = (cv) + (cw)$.

(f) Associativity of Scalar Multiplication: $c(dv) = (cd)v$.

(g) Unit for Scalar Multiplication: the scalar $1 \in \mathbb{R}$ satisfies $1v = v$. 