Def. Two subspaces \( W, Z \subset V \) are orthogonal if every vector in \( W \) is orthogonal to every vector in \( Z \).

It is enough to verify this on spanning sets (or bases) \( w_1, \ldots, w_k \) for \( W \) and \( z_1, \ldots, z_k \) for \( Z \): 

\[ \langle w_i, z_j \rangle = 0 \text{ for all } i, j. \]

Example: \( W = \{ \left( \frac{x}{2}, \frac{x}{3} \right) \mid x \in \mathbb{R} \} = \text{span} \left\{ \left( \frac{1}{2}, \frac{1}{3} \right) \right\} \)

\( Z = \text{span} \left\{ \left( -\frac{1}{6}, 0 \right), \left( -\frac{1}{3}, 1 \right) \right\} \)

\( w_1 \cdot z_1 = -2 + 2 + 0 = 0, \quad w_1 \cdot z_2 = -3 + 0 + 2 = 0. \) So \( W \perp Z \)

Def. The orthogonal complement of a subspace \( W \subset V \) is denoted by \( W^\perp \) and consists of all vectors orthogonal to \( W \).

Why is it the case that \( WN \cap W^\perp = \{ 0 \} \)? Why is it that \( W^\perp \) is a subspace? Note: \( W^\perp \) will be different for different inner products.

Example: For \( W \) above, what is \( W^\perp \)? We need \( \left( \frac{x}{2}, \frac{x}{3} \right) \in W^\perp \) if

\[ \langle \left( \frac{x}{2}, \frac{x}{3} \right), \left( \frac{1}{2}, \frac{1}{3} \right) \rangle = 0. \] 

Use Euclidean i.p. \( \Rightarrow \)

\[ x + 2y + 3z = 0 \] 

Homog linear system:

\[ x = -2y - 3z, \quad \left( \begin{array}{c} \frac{x}{2} \\ \frac{y}{3} \end{array} \right) = \left( \begin{array}{c} -2y - 3z \\ y \end{array} \right) = y \left( \begin{array}{c} -2 \\ 1 \end{array} \right) + z \left( \begin{array}{c} -3 \\ 0 \end{array} \right) \]

\( \Rightarrow W^\perp = \text{span} \{ W_1, W_2 \} \).
Theorem: Let $W \subset V$ be a finite dimensional subspace of an inner product space. Then every $v \in V$ can be uniquely decompose into $v = w + z$ where $w \in W$, $z \in W^\perp$.

(Illustrate geometrically!)

Proof: Let $w$ be the orthogonal projection of $v$ on $W$ and set $z = v - w$.

Uniqueness: If $v = w + z = w^* + z^*$, then $w - w^* = z - z^*$ and $w^* - w$ and $z^* - z$ are in both $W$ and $W^\perp$. Hence each is 0.

Example: Use $W$ and $W^\perp$ previously and $v = (\frac{1}{3}, 1)$. $w$ is the orthogonal projection on $W$, $z$ is the orthogonal projection on $W^\perp$ and each = $v$ - other. Which is easiest? ...

$w_1 = (\frac{1}{3})$, $w = \frac{<v, w_1>}{\|w_1\|^2}w_1 = \frac{1}{14}(\frac{1}{3})$,

$z = v - w = \frac{1}{14}(\frac{13}{3}, 2)$

Review Example 5.52 on transparency.

Proposition: If $W$ is finite dimensional, then $W^\perp W = W$.
EXAMPLE 5.52

Let \( W \subset \mathbb{R}^4 \) be the two-dimensional subspace spanned by the orthogonal vectors \( \mathbf{w}_1 = (1, 1, 0, 1)^T \) and \( \mathbf{w}_2 = (1, 1, 1, -2)^T \). Its orthogonal complement \( W^\perp \) (with respect to the Euclidean dot product) is the set of all vectors \( \mathbf{v} = (x, y, z, w)^T \) that satisfy the linear system

\[
\mathbf{v} \cdot \mathbf{w}_1 = x + y + w = 0, \quad \mathbf{v} \cdot \mathbf{w}_2 = x + y + z - 2w = 0.
\]

Applying the usual algorithm—the free variables are \( y \) and \( w \)—we find that the solution space is spanned by

\[
\mathbf{z}_1 = (-1, 1, 0, 0)^T, \quad \mathbf{z}_2 = (-1, 0, 3, 1)^T,
\]

which form a non-orthogonal basis for \( W^\perp \). An orthogonal basis

\[
\mathbf{y}_1 = \mathbf{z}_1 = (-1, 1, 0, 0)^T, \quad \mathbf{y}_2 = \mathbf{z}_2 - \frac{1}{2} \mathbf{z}_1 = (-\frac{1}{2}, -\frac{1}{2}, 3, 1)^T,
\]

for \( W^\perp \) is obtained by a single Gram–Schmidt step. To decompose the vector \( \mathbf{v} = (1, 0, 0, 0)^T = \mathbf{w} + \mathbf{z} \), say, we compute the two orthogonal projections:

\[
\mathbf{w} = \frac{1}{3} \mathbf{w}_1 + \frac{1}{7} \mathbf{w}_2 = \left( \frac{10}{21}, \frac{10}{21}, \frac{1}{7}, \frac{1}{21} \right)^T \in W,
\]

\[
\mathbf{z} = \mathbf{v} - \mathbf{w} = -\frac{1}{2} \mathbf{y}_1 - \frac{1}{21} \mathbf{y}_2 = \left( \frac{11}{21}, -\frac{10}{21}, -\frac{1}{7}, -\frac{1}{21} \right)^T \in W^\perp.
\]
Orthogonality of the Fundamental Subspaces

\[ Ax = b, \; A \in \mathbb{R}^{m \times n}, \; x \in \mathbb{R}^n, \; b \in \mathbb{R}^m \]

For this system to have a solution, \( b \) must lie in \( \text{rng } A \). If solutions are not unique then

\[ x = x^* + z, \; A x^* = b, \; z \in \text{ker } A, \; A z = 0. \]

Think of \( y = A x \) as defining a map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). Then we have the following geometry:

\[ \dim = n - r \]
\[ \dim = m - r \]

**Theorem:**

\[ \text{ker } A = (\text{corng } A)^\perp \]
\[ (\text{rng } A)^\perp = \text{coker } A \]

\[ 0 = A z = \begin{pmatrix} u_1^T \\ \vdots \\ u_m^T \end{pmatrix} z = \begin{pmatrix} u_1 \cdot z \\ \vdots \\ u_m \cdot z \end{pmatrix} \Rightarrow z \perp \text{ all rows of } A \]

\( A \) interchanges rows

Apply the same argument now to \( A^T \) to get the second statement.

**Theorem (Fredholm Alternative):** \( Ax = b \) has a solution iff \( b \) is \( \perp \) to the cokernel \( \text{coker } A \).
Ax = b has a soln iff \( y \cdot b = 0 \) for all solns of \( A^T y = 0 \)

Let \( y_1, \ldots, y_{m-r} \) be the a basis for \( \ker A \), then

\[
Ax = b \text{ has a soln iff } y_i \cdot b = 0, \; i = 1, \ldots, m-r
\]

Example: \( A = \begin{pmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{pmatrix} \). What restrictions on \( b \) ensure that \( Ax = b \) has a soln:

\[
\begin{bmatrix}
1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6
\end{bmatrix}
\begin{bmatrix}
b_1 \\ b_2 \\ b_3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
b_1 \\ b_2 \\ b_3-b_2-b_1
\end{bmatrix}
\]

So we need \( b_3 - b_2 - b_1 = 0 \)

What is the co-kernel of \( A \)?

\[
A^T = \begin{pmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ -7 & 5 & -2 \\ 9 & -3 & 6 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\rightarrow
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}
\]

\[
y \cdot b = \begin{pmatrix} -1 \\ 1 \\ b_2 \\ b_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = b_3 - b_2 - b_1 = 0
\]
Example.

Determine the compatibility conditions for

\[ x_1 - x_2 + 3x_3 = b_1 \]
\[ -x_1 + 2x_2 - 4x_3 = b_2 \]
\[ 2x_1 + 3x_2 + x_3 = b_3 \]
\[ x_1 + 2x_3 = b_4 \]

So first find a basis for \( \text{ker}(A) \)

\[
A^T = \begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 2 & 3 & 0 \\ 3 & -4 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -7z - 2w \\ -5z - w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -7 \\ -5 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}
\]

So we need

\[ y_1 \cdot b = -7b_1 - 5b_2 + b_3 = 0 \]
\[ y_2 \cdot b = -2b_1 - b_2 + b_4 = 0 \]
So the solution set of solutions is a translate of \( \ker A \).

We also see that one soln \( w \) has smallest norm. If \( \mathbf{z}_1, \ldots, \mathbf{z}_{n-r} \) form a basis of \( \ker A \), then \( w \) is the unique soln of \( A\mathbf{x} = \mathbf{b} \)

\[ \mathbf{z}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{z}_{n-r} \cdot \mathbf{x} = 0. \]

So this is a way of turning a system with inf.
many solns into one with exactly one soln!

The one with minimum norm

**Theorem.** Multiply by an \( m \times n \) matrix of rank \( r \) defines a one-to-one correspondence between the \( r \)-dimensional subspace \( \text{crg} A \subset \mathbb{R}^n \) and the \( r \)-dimensional subspace \( \text{rg} A \subset \mathbb{R}^m \). If \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) is a basis of \( \text{crg} A \), then \( A\mathbf{v}_1, \ldots, A\mathbf{v}_r \) is a basis of \( \text{rg} A \).
EXAMPLE 5.60

Consider the linear system

\[
\begin{pmatrix}
1 & -1 & 2 & -2 \\
0 & 1 & -2 & 1 \\
1 & 3 & -5 & 2 \\
5 & -1 & 9 & -6
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
1 \\
4 \\
6
\end{pmatrix}.
\]

(5.83)

Applying the usual Gaussian Elimination algorithm, we discover that the coefficient matrix has rank 3, and its kernel is spanned by the single vector \( z_1 = (1, -1, 0, 1)^T \). The system itself is compatible; indeed, the right hand side is orthogonal to the basis cokernel vector \( (2, 24, -7, 1)^T \), and so satisfies the Fredholm condition (5.81). The general solution to the linear system is \( x = (t, 3 - t, 1, t)^T \), where \( t = w \) is the free variable.

To find the solution of minimum Euclidean norm, we can apply the algorithm described in the previous paragraph.† Thus, we supplement the original system by the constraint

\[
\begin{pmatrix}
1 & -1 & 0 & 1 \\
x \\
y \\
z \\
w
\end{pmatrix}
= x - y + w = 0
\]

(5.84)

that the solution be orthogonal to the kernel basis vector. Solving the combined linear system (5.83, 84) leads to the unique solution \( x = w = (1, 2, 1, 1)^T \), obtained by setting the free variable \( t = 1 \). Let us check that its norm is indeed the smallest among all solutions to the original system:

\[
\|w\| = \sqrt{7} \leq \|x\| = \|(t, 3 - t, 1, t)^T\| = \sqrt{3t^2 - 6t + 10},
\]

where the quadratic function inside the square root achieves its minimum value at \( t = 1 \). It is further distinguished as the only solution that can be expressed as a linear combination of the rows of the coefficient matrix:

\[
w^T = (1, 2, 1, 1) \\
= -4 (1, -1, 2, -2) - 17 (0, 1, -2, 1) + 5 (1, 3, -5, 2).
\]