3.2 Inequalities

Cauchy-Schwarz Inequality.

\[ |<v,w>| \leq ||v|| ||w|| \quad \text{for all } v, w \in V \]

**Proof:** If \( w = 0 \), both sides are 0, so done.

Assume \( w \neq 0 \). Then

\[ 0 \leq ||v+tw||^2 = <v+tw, v+tw> = <v, v> + 2t<v, w> + t^2<w, w> \]

\[ = ||v||^2 + 2t<v, w> + t^2||w||^2 \]

Set \( p(t) = at^2 + bt + c \) where

\[ a = ||w||^2, \quad b = <v, w>, \quad c = ||v||^2 \]

So our inequality is

\[ 0 \leq p(t). \]

Where is the max of \( p \)?

\[ 0 = p'(t) = 2at + 2b \Rightarrow t = -\frac{b}{a} \]

So

\[ 0 \leq p\left(-\frac{b}{a}\right) = a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + c \]

\[ = \frac{b^2}{a} - 2\frac{b^2}{a} + c \]

\[ = c - \frac{b^2}{a} \]

\[ = \frac{ca - b^2}{a} = \frac{||v||^2 ||w||^2 - <v, w>^2}{||v|| ||w||} \]

We conclude that \( <v, w>^2 \leq ||v||^2 ||w||^2 \) and now take the square root.

Moreover, we have equality only if \( v \) and \( w \) are parallel.

We see that

\[ -1 \leq \frac{<v, w>}{||v|| ||w||} \leq 1, \quad \text{so we can define angle} \]

\( \theta \) between \( v \) and \( w \) by

\[ \cos \theta = \frac{<v, w>}{||v|| ||w||}. \quad \text{(Use cosine since } \theta = 0 \text{ corresponds to being parallel)}. \]
Def: Two vectors are orthogonal if \( \langle v, w \rangle = 0 \).

Example 1

\[ v = \left( \frac{1}{2} \right), \quad w = \left( \frac{6}{3} \right) \text{ on } \mathbb{R}^2 \]

\[ \langle v, w \rangle = 1 \cdot 6 + 2 \cdot (-3) = 6 - 6 = 0 \]

May not be orthog rel. to another inner product!

Example 2

\[ p(x) = x, \quad q(x) = x^2 - \frac{1}{2}, \quad \langle p, q \rangle = \int_{1}^{0} p(x)q(x) \, dx \]

So \[ \langle x, x^2 \rangle = \int_{0}^{1} x(x^2) \, dx = -\frac{1}{5} = 0 \]

Orthogonal polynomials!

Triangle Inequality

\[ \|v + w\| \leq \|v\| + \|w\| \]

Pf

\[ \|v + w\|^2 = \langle v + w, v + w \rangle \]

\[ = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle \]

\[ = \|v\|^2 + 2\langle v, w \rangle + \|w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \]

\[ = (\|v\| + \|w\|)^2 \quad \text{Done.} \]

What does this look like in certain cases?

\[ \left| \sum_{i=1}^{n} v_i w_i \right| \leq \sqrt{\sum_{i=1}^{n} v_i^2} \cdot \sqrt{\sum_{i=1}^{n} w_i^2} \]

...and for integrals.
Theorem. Every inner product \( \langle x, y \rangle \) on \( \mathbb{R}^n \) is given by
\[
\langle x, y \rangle = x^T K y
\]
for some positive definite real matrix \( K \).

Example
\[
K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}
\]
\[
g(x) = x^T K x = 4x_1^2 - 4x_1 x_2 + 3x_2^2
\]
\[
= \left( 4x_1^2 - 4x_1 x_2 + x_2^2 \right) + 2x_2^2
\]
\[
= (2x_1 - x_2)^2 + 2x_2^2.
\]
Thus \( g(x) > 0 \) unless \( x_2 = 0, 2x_1 = x_2 \Rightarrow x_1 = 0 \) i.e. \( x = 0 \).

Example
\[
K = \begin{pmatrix} a & b \\ b & a \end{pmatrix}
\]
\[
g(x) = ax_1^2 + 2bx_1 x_2 + cx_2^2
\]
\[
= a \left( x_1^2 + 2b \frac{x_1 x_2}{a} + \frac{b^2}{a^2} x_2^2 \right) + cx_2^2 - \frac{b^2}{a} x_2^2
\]
\[
= a \left( x_1 + \frac{b}{a} x_2 \right)^2 + \frac{ca-b^2}{a} x_2^2
\]
Thus \( g(x) \geq 0 \) iff \( a > 0 \) and \( ca - b^2 > 0 \)

\[
\Rightarrow \text{the (1,1) component AND positive determinant}
\]

Can also define positive semi-definite (i.e. \( g(x) = x^T A x \geq 0 \) \( \forall x \))
positive definite and negative semi-definite for symmetric matrices and quadratic forms.

Null Directions!
Now for a warning. Keep your wits about you. Consider the following inner product on $\mathbb{R}^2$:

$$\langle x, y \rangle = x^T \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} y$$

$$= (x_1, x_2) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= (x_1 - x_2, -x_1 + 2x_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2.$$

So what is the angle $\theta$ here? Well:

$$\langle x, y \rangle = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 \\ 0 \end{pmatrix} = 0 \times 1 + 1 \times 0$$

Thus $x$ and $y$ are orthogonal, and so $\theta = 90^\circ$!

Everything now is relative to some inner product!