Dot product: \[
    \langle v, w \rangle = v \cdot w = v_1 w_1 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i = v^T w
\]
\[
    || v || = \sqrt{\langle v, v \rangle} = \sqrt{v_1^2 + \cdots + v_n^2}
\]

**Defn.** An inner product is a map from \( V \times V \) to \( \mathbb{R} \) satisfying:

- **Bilinearity**
  a) \( \langle cu + dv, w \rangle = c \langle u, w \rangle + d \langle v, w \rangle \) for all \( c,d \in \mathbb{R} \)
  b) \( \langle u, cv + dw \rangle = c \langle u, v \rangle + d \langle u, w \rangle \)

- **Symmetry**
  a) \( \langle u, v \rangle = \langle v, u \rangle \)

- **Positivity**
  a) \( \langle v, v \rangle > 0 \) for all \( v \neq 0 \), \( \langle 0, 0 \rangle = 0 \)

The norm corresponding to \( \langle , \rangle \) is
\[
    || v || = \sqrt{\langle v, v \rangle}.
\]

**Examples:** In \( \mathbb{R}^2 \), define \( \langle v, w \rangle = 2v_1 w_1 + 5v_2 w_2 \)

- weighted Euclidean.

\[
    || v ||^2 = \langle v, v \rangle = 2v_1^2 + 5v_2^2 > 0 \text{ unless } v_1 = v_2 = 0.
\]

Certainly linear in \( v_1 \) and in \( w_1 \). And symmetric. Why would this norm and inner product be useful?

**Example** \( \langle v, w \rangle = v_1 w_1 - v_1 w_2 - 2v_2 w_1 + 4v_2 w_2 \)

\[
    || v ||^2 = \langle v, v \rangle = v_1^2 - 2v_1 v_2 + 4v_2^2 = (v_1 - v_2)^2 + 3v_2^2 \geq 0
\]

etc.
Function Spaces:

\[ C^0[a, b] = \text{space of continuous real valued functions on } [a, b]. \]

\[ \langle f, g \rangle = \int_a^b f(x)g(x) \, dx. \]

Check linearity and symmetry.

\[ \| f \| = \sqrt{\int_a^b f(x)^2 \, dx} \]

when is this 0?

Example: On \([a, b] = [0, \pi/2]\) we have

\[ \langle \sin x, \cos x \rangle = \int_0^{\pi/2} \sin x \cos x \, dx = \frac{1}{2} \sin^2 \frac{\pi}{2} \]

\[ = \frac{1}{2} \]

\[ \| \sin x \| = \sqrt{\int_0^{\pi/2} \sin^2 x \, dx} = \sqrt{\frac{\pi}{4}} \]

\[ \| 1 \| = \sqrt{\int_0^{\pi/2} 1^2 \, dx} = \sqrt{\frac{\pi}{2}}. \]