1. (6 points) Let $V$ be $C^0[0, 1]$, the vector space of real-valued functions that are defined and continuous on $[0, 1]$. Show that the subset

$$W = \{ f \in V | f(0) = f(1) \}$$

is a subspace of $V$.

Answer: Let $f$ and $g$ lie in $W$. This means that $f$ and $g$ are continuous and $f(0) = f(1)$ and $g(0) = g(1)$. Then for any real numbers $c$ and $d$

$$(cf + dg)(0) = cf(0) + dg(0)$$
$$= cf(1) + dg(1)$$
$$= (cf + dg)(1)$$

In addition, $cf + dg$ is also a continuous function and so $cf + dg$ lies once again in $W$. Thus $W$ is a subspace of $V$.

2. (12 points) A certain digraph has the incident matrix

$$A = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

(a) Draw a digraph that corresponds to this matrix.

Answer: (3 points)
(b) What is a basis for \( \ker A \)?

Answer: (1 points) The single vector \( \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T \) is a basis for the kernel (proved in class).

(c) By finding a basis for one of the other fundamental subspaces of \( A \), find the independent circuits in this digraph.

Answer: (8 points) In this case we need to find a basis of the cokernel of \( A \):

\[
A^T = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 \\
0 & -1 & 0 & 0 & 0 & -1 \\
\end{pmatrix} \to \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 1 & 1 & 0 & -1 \\
\end{pmatrix} \to \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 1 & 1 & 0 & -1 \\
\end{pmatrix} \to \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

If our variables are \( x, y, z, u, v, w \), then \( u = -v + w, z = v, y = -z - u = -v - (-v + w) = -w, x = -y = w \) so

\[
\begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} w \\ -w \\ v \\ -v + w \\ v \\ w \end{pmatrix} = v \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

Thus the independent circuits are (3)+(5)-(4) and (1)+(4)+(6)-(2).

3. (12 points) Let

\[
\langle x, y \rangle = x^T K y \quad \text{where} \quad K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}
\]

(a) Verify that \( \langle x, y \rangle \) given above defines a valid inner product on \( \mathbb{R}^3 \).

Answer: (6 points) In class we showed that all inner products on \( \mathbb{R}^3 \) have the form \( x^T K y \) provided that \( K \) is symmetric and positive definite. To show that \( K \) is positive definite use Gaussian Elimination:

\[
K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}
\]

Since all pivots are positive, \( K \) is positive definite.
(b) Find a basis for the set of all vectors that are orthogonal to \( \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T \) in terms of the inner product given above.

Answer: (6 points) Setting \( \mathbf{x} = (x \ y \ z)^T \), we need

\[
0 = \langle \mathbf{x}, (1 \ 2 \ 1)^T \rangle = (x \ y \ z) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 2y
\]

Thus \( y = 0 \) and \( x \) and \( z \) are free variables, so

\[
\mathbf{x} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\]

and so \( e_1 \) and \( e_3 \) are a basis

4. (10 points) Find the least squares solution to the following system:

\[
\begin{align*}
2x + y &= 1 \\
x - y &= 2 \\
x + 5y &= 3
\end{align*}
\]

Answer: Set

\[
A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]

Then we need to solve \( Kx = f \) where

\[
K = A^T A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix}^T \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 27 \end{pmatrix}
\]

\[
f = A^T b = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 5 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}.
\]

Thus

\[
\begin{pmatrix} 6 & 6 & 6 \\ 6 & 27 & 14 \end{pmatrix} \rightarrow \begin{pmatrix} 6 & 6 & 7 \\ 0 & 21 & 7 \end{pmatrix}
\]

Thus \( y = \frac{1}{3} \) and \( x = \frac{7}{6} - \frac{1}{3} = \frac{5}{6} \) and these are the coordinates of the least squares solution.

5. (10 points) Find all vectors in \( P(3) \) that are orthogonal to \( p_1(x) = 1 \) and \( p_2(x) = x \) in the inner product

\[
\langle p, q \rangle = \int_{-1}^{1} f(x)g(x)dx
\]

Answer: Begin with a general vector from \( P(3) \), say \( q(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \).
Then we need
\[ 0 = \int_{-1}^{1} (a_0 + a_1 x + a_2 x^2 + a_3 x^3) \, dx \]
\[ = 2a_0 + 0a_1 + \frac{2}{3}a_2 + 0a_3 \]
\[ 0 = \int_{-1}^{1} x(a_0 + a_1 x + a_2 x^2 + a_3 x^3) \, dx \]
\[ = 0a_0 + \frac{2}{3}a_1 + 0a_2 + \frac{2}{5}a_3 \]

We conclude that \( a_2 \) and \( a_3 \) are free variables and \( a_0 = -\frac{1}{3}a_2 \), \( a_1 = -\frac{3}{5}a_3 \). So the general orthogonal vector is
\[
q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
= a_2 (x^2 - \frac{1}{3}) + a_3 (x^3 - \frac{3}{5}x)
\]

6. (10 points)

(a) An \( m \times n \) matrix \( A \) has rank \( r \). Which of the following statements is correct?
   i. \( A \) has \( r \) linearly independent columns
   ii. \( A \) has \( r \) linearly independent rows
   iii. Both i) and ii) are true
   iv. Neither i) or ii) is true
   Choice: iii

(b) If the incident matrix for a connected digraph is \( n \times n \), how many independent circuits are there in the digraph?
   i. 0
   ii. 1
   iii. more than 1
   iv. impossible to determine without further information
   Choice: ii

(c) If an \( m \times n \) matrix \( A \) has rank \( r \), then the dimension of ker \( A \) is
   i. \( r \)
   ii. \( n - r \)
   iii. \( m - r \)
   iv. \( m + n - 2r \)
   Choice: ii
(d) Here are two statements:

Statement A : If $x_1$ and $x_2$ are both solutions of $Ax = 0$,
then so is $c_1x_1 + c_1x_2$ for any real numbers $c_1$ and $c_2$

Statement B : ker $A$ is a subspace

Which of the following options is the case:

i. Neither statement implies the other
ii. Statement A implies Statement B but B does not imply A
iii. Statement B implies Statement A but A does not imply B
iv. The two statements are equivalent

Choice: iv

(e) In the representation $x = x^* + z$ of solutions of $Ax = b$,

i. $x - x^*$ satisfies the homogeneous system
ii. $x^*$ is not unique if ker $A$ is non-trivial
iii. both i) and ii) are true
iv. neither i) nor ii) is true

Choice: iii