Math 415 Old Exam # 1

1. (12 points)
   (a) Find an $LU$ factorization of the matrix
   \[
   A = \begin{pmatrix}
   3 & 2 & 0 \\
   3 & 1 & -2 \\
   -6 & -7 & -5
   \end{pmatrix}
   \]
   (b) If a matrix $B$ has the $LU$ factorization
   \[
   B = \begin{pmatrix}
   2 & 2 & -5 \\
   -8 & -7 & 18 \\
   0 & 3 & -7
   \end{pmatrix} = \begin{pmatrix}
   1 & 0 & 0 \\
   -4 & 1 & 0 \\
   0 & 3 & 1
   \end{pmatrix} \begin{pmatrix}
   2 & 2 & -5 \\
   0 & 1 & -2 \\
   0 & 0 & -1
   \end{pmatrix}
   \]
   solve $Bx = \begin{pmatrix} -3 \\ 10 \\ -7 \end{pmatrix}^T$ without reducing the augmented matrix.
   (c) What is the value of $\det B$?

2. (12 points)
   (a) Give the definition of the inverse of a matrix $A$.
   (b) Write down a $3 \times 3$ permutation matrix $P$. Also write down its inverse.
   (c) Write down a $3 \times 3$ matrix $E$ that performs an elementary row operation through matrix multiplication. Also write down its inverse.
   (d) Compute explicitly the inverse of the product $PE$.
   (e) What does the value of the determinant of a $3 \times 5$ matrix tell you about the existence of an inverse of that matrix?

3. (8 points)
   (a) Give a definition of the rank of a matrix $A$.
   (b) Find the rank of the matrix
   \[
   A = \begin{pmatrix}
   0 & 1 & 2 \\
   0 & 2 & 4 \\
   0 & 3 & 6 \\
   1 & 4 & 8
   \end{pmatrix}
   \]
   (c) What is the rank of $A^T$?
4. (12 points)

(a) Give a definition of the concept of linear independence of a collection of vectors \( \{v_1, v_2, \ldots, v_n\} \) from a vector space \( V \).

(b) For \( V = M_{2 \times 2} \) show that
\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\end{align*}
\]
are linearly independent.

5. (12 points)

(a) What is meant by the span of a collection of vectors \( \{v_1, v_2, \ldots, v_n\} \) from a vector space \( V \)?

(b) Show that all vectors in the span of the two vectors
\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}
\end{align*}
\]
lie in the plane in \( \mathbb{R}^3 \) defined by \( x + 2y + 3z = 0 \).

(c) What is the dimension of the subspace
\[
W = \left\{ (x, y, z)^T \text{ for which } x + 2y + 3z = 0 \right\}
\]
given the facts outlined in part b)? Be specific.

6. (10 points) The three vectors
\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\end{align*}
\]
form a basis for \( \mathbb{R}^3 \) (you do NOT need to verify this!)

(a) Express the vector \( u = \begin{pmatrix} 5 & -8 & -1 \end{pmatrix}^T \) as a linear combination of these three vectors.

(b) What are the “coordinates” of \( u \) relative to this basis?