Applications of the Integral Theorems

Two features: mass density \( \rho(t,x,y,z) \) \( \frac{\text{mass}}{\text{vol}} \)
fluid velocity \( v(t,x,y,z) \) \( \text{3D vector field} \)

How are these related?

Consider an arbitrary region \( E \) in the fluid, with (positively oriented) boundary surface \( S \).

Balance Law: \( \frac{d}{dt} \iint_E \rho \, dV = \iiint_S \rho v \cdot n \, dS \)
\( \frac{\text{total mass in } E}{\text{total mass in } E} \) \( \frac{\text{net flow of mass out of } E}{\text{net flow of mass into } E} \)
\( \frac{\text{increase in total mass}}{\text{net flow of mass into } E} \)

\[ \iiint_E \frac{\partial \rho}{\partial t} \, dV + \iint_S \rho v \cdot n \, dS = 0 \]
Divergence Thm

\[ \iiint_E \text{div}(\rho v) \, dV \]

\[ \iiint_E \left( \frac{\partial \rho}{\partial t} + \text{div}(\rho v) \right) \, dV = 0 \]

But \( E \) is arbitrary, so (explain)
\( \frac{\partial \rho}{\partial t} + \text{div}(\rho v) = 0 \) Continuity Equation

If \( \rho \) is constant and time-independent (water has this feature),
we get then \( \text{div} \, v = 0 \), incompressibility condition.
Electrostatics. Recall that charges:

\[ F = \frac{\epsilon q Q}{r^3} \]

The force \( q \) exerts on \( Q \). The force is conservative!

Electric Field (due to \( q \)):

\[ E = \frac{1}{q} F = \frac{q Q}{r^3} \]

How does this generalize if we have instead an electrically charged gel (say) with charge density \( q(x,y,z) \). This produces an electric field \( E(x,y,z) \). Then \( qE \) is the force felt by a point charge \( Q \) placed at \( (x,y,z) \).

Balance Equation: \( \epsilon \iiint q \, dV = \iiint E \cdot n \, dS \)

(Gauss' Law)

more charge in \( E \) \( \Rightarrow \) more flux of \( E \) out of \( S \)

\[ \iiint q \, dV = \iiint E \cdot n \, dS = \iiint \text{div} E \, dV \]

\[ \implies \iiint (q - \text{div} E) \, dV = 0 \text{ with } E \text{ an arbitrary region} \]

\[ \implies \text{div} E = q \text{ at each point } (x,y,z) \]

This is one of Maxwell's Equations of Electromagnetic Theory.

In electrostatics you can show that \( E \) is a conservative field, so \( E = \nabla \phi \) for some scalar \( \phi \). Thus

\[ \text{div} E = \text{div} \nabla \phi = \Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = q(x,y,z) \text{. Poisson's Eqtn.} \]

Given \( q \), solve for \( \phi(x,y,z) \)