Lecture 27 Sec 15.10

\[ \int_a^b f(x) \, dx = \int_c^d f(g(u)) \, du \quad \text{for invertible } u = g(x), \quad \text{constant } a, b, \quad \text{constant } \alpha, \beta \]

What is the generalization to double and triple integrals?

\[ T(u, v) = (g(u, v), h(u, v)) = (x, y) \]

Assume \( T \) is one-to-one meaning? For each \((x, y)\), only one \((u, v)\)

\[ T^{-1}(x, y) = (g(x, y), h(x, y)) = (u, v) \]

**Ex.** \( x = u^2 - v^2 \), \( y = 2uv \), \( S = \{ (u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1 \} \)

- \( S_1: v = 0, 0 \leq u \leq 1 \)
  \[ \Rightarrow x = u^2, \quad y = 0, \quad 0 \leq u \leq 1 \]
- \( S_2: u = 1, 0 \leq v \leq 1 \)
  \[ \Rightarrow x = 1 - u^2, \quad y = 2v \Rightarrow x = 1 - \frac{v^2}{4} \quad \text{parabola} \]
  \( x \) goes from 1 to 0 along \( y \)
- \( S_3: v = 1, 0 \leq u \leq 1 \)
  \[ \Rightarrow x = u^2 - 1, \quad y = 2u \Rightarrow x = \frac{y^2}{4} - 1 \]
  \( x \) goes from 0 to -1 as \( u \) goes from 1 to 0
- \( S_4: u = 0, 0 \leq v \leq 1 \)
  \[ \Rightarrow x = -v^2, \quad y = 0 \]
  \( x \) goes from -1 to 0 as \( v \) goes from 1 to 0
\[
\Delta A = \Delta u \Delta v \quad \text{After all, } u \text{ and } v \text{ could be angles!}
\]

\[
\langle x, y \rangle = \langle r(u, v) = \langle g(u, v), h(u, v) \rangle
\]

\text{Area of } R_{ij} \text{ is approximated by area of rectangle parallelog...}

\text{Recall:}

\[
|a \times b| = |a| |b| \sin \theta = |a \times b|
\]

\[
\text{Area of } \frac{1}{2} R_{ij} = |\Gamma_u \Delta u \times \Gamma_v \Delta v| = |\Gamma_u \times \Gamma_v| \Delta u \Delta v
\]

\[
\Delta A = A(R_{ij})
\]

\[
\Gamma_u \times \Gamma_v = \langle x_u, y_u, 0 \rangle \times \langle x_v, y_v, 0 \rangle = \langle y_u \cdot 0 - 0 \cdot y_v, 0 \cdot x_v - x_u \cdot 0, x_u \cdot y_v - y_u \cdot x_v \rangle
\]

So \[
|\Gamma_u \times \Gamma_v| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| = \left| \frac{\partial (x, y)}{\partial (u, v)} \right| = \left| \frac{\partial^2 y}{\partial u \partial v} - \frac{\partial^2 x}{\partial u \partial v} \right| = \text{Jacobian of } x, y \text{ wrt } u, v
\]

\[
dA = dx \, dy = \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, du \, dv
\]
\[ x = r \cos \theta, \quad y = r \sin \theta \]

\[
\begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r
\]

so
\[ dA = dx \, dy = r \, dr \, d\theta \]