Rework the box problem using Lagrange multipliers
Talk about level surfaces now!

Ex. Compare
\[ \max f = x^2 + 2y^2 \quad \text{to} \quad \max f = x^2 + 2y^2 \]
\[ \text{when } x^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 \leq 1 \]
\[ f(\pm 1, 0) = 1, \quad f(0, \pm 1) = 2 \quad \text{both sets are closed and bounded} \]

Two Constraints:
Find max and min of \( f(x, y, z) \) subject to
\[ g(x, y, z) = k, \quad h(x, y, z) = c \]
these intersect in a curve
now picture a level surface of \( f \) just "kissing" this curve in a point.
at this point, how are \( \nabla f, \nabla g, \nabla h \) related?

\[ \nabla f = \lambda \nabla g + \mu \nabla h \quad \text{at the max/min point} \]
Ex. Find max. of \( f(x,y,z) = x + 2y + 3z \) subject to:

\[
g(x,y,z) = x - y + z = 1, \quad h(x,y,z) = x^2 + y^2 = 1
\]

plane with normal \( \langle 1, -1, 1 \rangle \)

\[\nabla f = \lambda \nabla g + \mu \nabla h
\]

\[
1 = f_x = \lambda g_x + \mu h_x = \lambda + 2\mu \quad \rightarrow \quad 2\mu - \lambda = -2
\]

\[
2 = f_y = \lambda g_y + \mu h_y = -\lambda + 2\mu \quad \rightarrow \quad 2\mu + \lambda = 5
\]

\[
3 = f_z = \lambda g_z + \mu h_z = \lambda
\]

\[
x - y + z = 1, \quad x^2 + y^2 = 1
\]

\[
\Rightarrow \quad x = \frac{2\mu}{5}, \quad y = \frac{-2}{5} \quad \rightarrow \quad 4x^2 + (2y)^2 = 4
\]

\[
x^2 + (5x)^2 = 4
\]

\[
z = 1 - x + y
\]

\[
z = 1 + \frac{\sqrt{29}}{\sqrt{29}} + \frac{5}{\sqrt{29}} = \frac{\sqrt{29} - 7}{\sqrt{29}} + \frac{\sqrt{29} + 7}{\sqrt{29}}
\]

\[
f\left( \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, \frac{\sqrt{29} - 7}{\sqrt{29}} \right) = \frac{2}{\sqrt{29}} - \frac{10}{\sqrt{29}} + \frac{3\sqrt{29} - 21}{\sqrt{29}} = 3 - \sqrt{29} \leq \text{min}
\]

\[
f\left( \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, \frac{\sqrt{29} + 7}{\sqrt{29}} \right) = \frac{-2}{\sqrt{29}} + \frac{10}{\sqrt{29}} + \frac{3\sqrt{29} + 21}{\sqrt{29}} = 3 + \sqrt{29} \leq \text{max}
\]