Absolute Max and Min Values.

A set is closed if it contains all its boundary pts

A set is bounded if it can be contained inside a large enough ball

Theorem: If $f$ is continuous on a closed and bounded set $D$ in $\mathbb{R}^2$, then $f$ has an absolute max and an absolute min over the points in $D$.

Procedure: Find values of $f$ at all critical pts in $D$, then find extreme values of $f$ on the boundary of $D$, and choose the largest and smallest values from these.

Example: $f(x,y) = x^2 - 2xy + 2y$ on $D = \{(x,y) | 0 \leq x \leq 3, 0 \leq y \leq 2\}$

Inside $D$: $0 = f_x = 2x - 2y$, $0 = f_y = -2x + 2$

$\Rightarrow (x,y) = (1,1)$, $f(1,1) = 1$

On $L_1$: $f(x,0) = x^2$, $0 \leq x \leq 3 \Rightarrow 0 \leq f(x,0) \leq 9 = f(3,0)$

On $L_2$: $f(x,2) = x^2 - 4x + 4 = (x-2)^2 \Rightarrow 0 = f(2,2) \leq f(x,y) \leq f(0,2) = 4$

On $L_3$: $f(3,y) = 9 - 4y$, $0 \leq y \leq 2 \Rightarrow 1 = f(3,2) \leq f(x,y) \leq f(3,0) = 9$

On $L_4$: $f(0,y) = 2y$, $0 = f(0,0) \leq f(x,y) \leq f(0,2) = 4$

$\Rightarrow$ Abs max is 9 at (3,0), abs min is 0 at (0,0) and (2,2)
Sec 14.8

Constrained Optimization - Lagrange Multipliers

Find the extreme values of \( f(x,y) \) subject to \( g(x,y) = k \)

\[ \nabla f \text{ and } \nabla g \text{ are in same direction (or at least parallel).} \]

Procedure: Assume \( \nabla g \neq 0 \) along \( g(x,y) = k \)

1. Find all values of \( x, y, \lambda \) where
   \[ g(x,y) = k \]
   \[ \nabla f(x,y) = \lambda \nabla g(x,y) \]
   \[ \{ \text{Begin, 3 unknowns} \} \]

2. Find value of \( f \) at each point found. The largest is a local max and the smallest is a local min.

Ex \[ f(x,y) = x^2 + 2y^2 \text{ on } x^2 + y^2 = 1 \text{ (absolute max and min why?)} \]

\[ \Rightarrow x^2 + y^2 = 1 \]
\[ \Rightarrow 2x = \lambda g_x = 2x \lambda \Rightarrow x (\lambda - 1) = 0 \]
\[ \Rightarrow 4y = \lambda g_y = 2y \lambda \Rightarrow y (\lambda - 2) = 0 \]

\( x \) and \( y \) cannot both be zero (why), so

\[ x = 0 \text{ and } \lambda = 2, \quad y = 0, \lambda = 1 \]

\[ \Rightarrow x^2 + y^2 = 1 \]
\[ \Rightarrow x^2 + 0^2 = 1 \]

\( x,y = (0,0), (1,0), (-1,0) \)

\[ f(0,0) = 2, f(0,-1) = 2 \]

\( f(1,0) = 1, f(-1,0) = 1 \)
Ex. Rectangular box revisited

Maximize $V = xyz$ subject to $g(x, y, z) = 2xz + 2yz + xy = 12$

$V_x = \lambda g_x, V_y = \lambda g_y, V_z = \lambda g_z, \lambda = 12$

\[
\begin{align*}
xyz &= \lambda (2z + y), & xz &= \lambda (2z + x), & xy &= \lambda (2x + y),
\end{align*}
\]

Note $\lambda > 0$ (why?), so

\[
\begin{align*}
2xz + xy &= 2yz + xy = 2z x + 2z y
\end{align*}
\]

$x = y, 2z = x$

\[
\begin{align*}
2(xz)(z) + 2(yz)(z) + (xz)(xz) &= 12 \\
12z^2 &= 12, z = 1
\end{align*}
\]

$x = 2, y = 2, z = 1$