**Definition:** 
(a, b) is a critical point of \( f(x, y) \) if \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \), or one of \( f_x \) and \( f_y \) do not exist at \((a, b)\).

**Theorem:** The local maxima and minima of \( f \) occur at critical points.

\[
\begin{align*}
\text{Ex:} & \quad f(x, y) = x^2 + y^2 - 2x - 6y + 14 \\
f_x &= 2x - 2, \quad f_y = 2y - 6 \quad \text{and both exist everywhere} \\
\text{So} \quad 0 &= 2x - 2, \quad 0 = 2y - 6 \quad \Rightarrow \quad (x, y) = (1, 3) \\
\text{Note:} & \quad f(x, y) = (x-1)^2 + (y-3)^2 + 4 \\
\text{So} \quad f(x, y) \geq f(1, 3) = 4. \quad \text{(Global Minimum)}
\end{align*}
\]

\[
\begin{align*}
\text{Ex:} & \quad f(x, y) = y^2 - x^2, \quad 0 = f_x = -2x, \quad 0 = f_y = 2y \quad \Rightarrow \quad (x, y) = (0, 0) \\
\text{But} \quad f(x, 0) = -x^2 \quad \text{concave up in} \ x \ \text{direction} \\
\quad f(0, y) = y^2 \quad \text{concave down in} \ y \ \text{direction} \\
\text{saddle pt}
\end{align*}
\]

**Second Deriv Test:** At the very minimum we need \( f_{xx} > 0 \) at a local minimum (why geometrically?)

Let the second deriv of \( f \) be continuous around \((a, b)\) and \( f_x(a, b) = 0 = f_y(a, b) \). Set

\[
D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2
\]

Then

1. If \( D > 0 \) and \( f_{xx}(a, b) > 0 \) \( \Rightarrow \) local min at \((a, b)\)
2. If \( D > 0 \) and \( f_{xx}(a, b) < 0 \) \( \Rightarrow \) local max at \((a, b)\)
3. If \( D < 0 \) \( \Rightarrow \) a saddle pt at \((a, b)\)
4. If \( D = 0 \) \( \Rightarrow \) inconclusive
Ps. For direction \( u = \langle h, k \rangle \), 
\[ D_u f = f_x h + f_y k \]
unit vector \( \hat{u} \):
\[ D_u^2 f = D_u (D_u f) = (f_x h + f_y k)^2 h + f_y k \]
\[ = f_{xx} h^2 + 2 f_{xy} h k + f_{yy} k^2 \]

(complete square):
\[ f_{xx} \left( h + \frac{f_{xy}}{f_{xx}} k \right)^2 + \frac{k^2}{f_{xx}} (f_{xx} f_{yy} - f_{xy}^2) \]

Now discuss how our theorem follows.

Ex. Find all local max, mins, and saddle points of
\[ f(x, y) = x^4 + y^4 - 4xy + 1 \]
0 = \( f_x = 4x^3 - 4y \) \( f_y = 4y^3 - 4x \)
y = \( x^3 \) and \( x = \frac{y}{2} \)
\[ z = (x^3)^2 \] or \( x (x^3 - 1) = 0 \)
\[ x(x-1)(x+1)(6x^2+1)(x^4+1) = 0 \]
so \( x = 0, y = 0 \) or \( x = 1, y = 1 \) or \( x = -1, y = -1 \).

\[ f_{xx} = - \quad f_{yy} = - \quad f_{xy} = - \]
\[ D = \ldots = 144x^2y^2 - 16 \]

Ex. Find shortest distance from \((0, -2)\) to \( x + 2y + z = 4 \)
We are denoting any pt on plane by \((x, y, z)\), so distance is
\[ d^2 = (x-1)^2 + (y+2)^2 + (z+2)^2 \]
and we can just minimize \( d^2 = f(x, y, z) \). But \( z = 4 - x - 2y \)
So \( d^2 = f(x, y) = (x-1)^2 + y^2 + (4-x-2y+2)^2 \)
\[ f_x = \ldots = 4x + 4y \]
\[ 14 = 0 \]
\[ f_y = \ldots = 4x + 4y \]
\[ 10y - 24 = 0 \]
Solve simultaneously:
\[ (x, y) = (\frac{11}{6}, \frac{5}{3}) \]
Now show \( f_{xx} = 4 > 0 \), \( D = 24 > 0 \) \( f(\frac{11}{6}, \frac{5}{3}) = \frac{25}{6} \) local min.
Ex: Rectangular box made from 12 m$^2$ of cardboard (no lid). What is max volume?

\[ V = xyz \]

\[ 2yz + 2xz + xy = 12 \]

\[ z = \frac{12-xy}{2(x+y)} \]

So \[ V = f(x,y) = xy \frac{(12-xy)}{2(x+y)} \]

Now find \( f_x = 0, f_y = 0 \)

\[ f_x = y^2 \frac{(12-2xy-x^2)}{2(x+y)^2}, \quad f_y = x^2 \frac{(12-2xy-y^2)}{2(x+y)^2} \]

so \[ 12-2xy-x^2 = 0 = 12-2xy-y^2 \]

so \[ x^2 = y^2 \], so \( x = y \) (we are only interested in \( x > 0,y > 0 \))

so \[ 12-2x^2-x^2 = 0 \], or \[ x^2 = 4 \] or \( x = y = 2 \) \( \Rightarrow z = 1 \)

\[ V = 4 \text{ m}^3 \]