Last day we saw how to compute partial derivatives. Now let's interpret what we were finding.

Since
\[ f_x(a, b) = \frac{\partial}{\partial x} (f(x,b)) \bigg|_{x=a} \]

This is the slope of the trace of the surface on \( y=b \).

Similarly the value \( f_y(a, b) \) is the slope of the trace of the surface on \( x=a \).

So \( f_x \) and \( f_y \) measure slope of traces parallel to \( x=\text{const} \) and \( y=\text{const} \). What about traces along \( x=y \) say?

Demonstrate a surface with no tangent along \( x=y \) or \( x=-y \). This shows \( f_x \) and \( f_y \) contain limited information!

Ex: Implicit differentiation:
\[ x^3 + y^3 + z^3 + 6xyz = 1 \]

Find \( Z_x \) and \( Z_y \) with \( Z=Z(x,y) \) defined implicitly.

\[ \frac{\partial}{\partial x} \text{ eqn } = 3x^2 + 3z^2 Z_x + 6yz + 6xy Z_x = 0 \]

\[ \text{Solve for } Z_x = Z_y \]

\[ \frac{\partial}{\partial y} \text{ eqn } = 3y^2 + 3z^2 Z_y + 6xz + 6xy Z_y = 0 \]
Higher Derivatives: \( f_{xx} = \frac{\partial^2 f}{\partial x^2} = (f_x)_x, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = (f_x)_y, \) etc.

 Clairaut's Theorem: Let \( f: \mathbb{D} \to \mathbb{R} \) with \((a,b)\) in \( \mathbb{D} \). If \( f_{xy} \) and \( f_{yx} \) are continuous in \( \mathbb{D} \), then
\[ f_{xy}(a,b) = f_{yx}(a,b). \]

Sec 14.4

**Tangent planes.** The tangent plane to \( z = f(x,y) \) at \((a,b)\) is the plane through \((a,b)\) determined by

\[ T_1: \text{tangent line at } (a,b) \text{ in the trace } x=a \]

and \( T_2: \) the tangent line at \((a,b)\) to the trace \( y=b \)

Let \( x_0 = a, \ y_0 = b, \ z_0 = f(a,b) \). Then
\[ A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \]
or
\[ z-z_0 = P(x-x_0) + Q(y-y_0) \quad \text{with } \quad P = -\frac{A}{C}, \ Q = -\frac{B}{C} \]

Set \( y = y_0 \Rightarrow z-z_0 = P(x-x_0) + Q(y-y_0) \) in \( T_2 \) so \( P = f_x(x_0,y_0) \)

Similarly: \( Q = f_y(x_0,y_0) \)

ie, the tangent plane is
\[ z-z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \]

**Ex:** What is the tangent plane \( \pi \) to \( z = 2x^2 + y^2 \) at \((1,1,3)\)

\[ f_x = 4x, \ f_y = 2y, \ f_x(1,1) = 4, \ f_y(1,1) = 2 \]

\[ \Rightarrow z-3 = 4(x-1) + 2(y-1) \]

Linear Approximations

\[ f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \text{ is the linear approx to } f(x,y) \text{ for } (x,y) \text{ near } (a,b) \]
$f(x,y) = xe^{xy}$ Find linear approx. near $(1,0)$ and use it to estimate $f(1.1, -0.1)$

$f_x(x,y) = xe^{xy} + xy e^{xy}, \quad f_y(x,y) = x^2 e^{xy}$

$f_x(1,0) = 1, \quad f_y(1,0) = 1$

$L(x,y) = f(1,0) + (x-1) + (y-0)$

$= 1 + x-1 + y = x + y$

$L(1.1, -0.1) = 1.1 - 0.1 = 1$

Actually $f(1.1, -0.1) = 1.1 e^{-0.1} = 0.98542$

Note: Tangent planes are not always good approximations!

Defin: $z = f(x,y)$ is differentiable at $(a,b)$ if

$\left| f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b) \right|$

$= | x-1 | + | y-0 |$

where $e_1$ and $e_2 \to 0$ as $(x,y) \to (a,b)$

Theorem: If $f_x$ and $f_y$ exist and are continuous at $(a,b)$, then $f$ is differentiable at $(a,b)$