Common limits: \( \lim_{(x,y) \to (a,b)} x = a \), \( \lim_{(x,y) \to (a,b)} y = b \), \( \lim_{(x,y) \to (a,b)} C = C \)

\( \lim_{(x,y) \to (a,b)} \text{sum} = \text{sum} \lim_{(x,y) \to (a,b)} \text{prod} = \text{prod} \lim_{(x,y) \to (a,b)} \)

\( \lim_{(x,y) \to (a,b)} (f(x,y)) = \lim_{(x,y) \to (a,b)} f \), \( \lim_{(x,y) \to (a,b)} f = \lim_{(x,y) \to (a,b)} g \) if \( \lim_{x \to a} g = 0 \)

\( \lim_{(x,y) \to (a,b)} h(f(x,y)) = h(\lim_{(x,y) \to (a,b)} f) \) if \( h \) is continuous

Example: \( \lim_{(x,y) \to (0,0)} \frac{x^2+y^2-1}{x+y} = \lim_{(x,y) \to (0,0)} \frac{x^2+y^2-1}{x+y} = \frac{0^2+0^2-1}{0+0} = -1 \)

In general, indeterminate forms \( \frac{0}{0} \) are the difficult ones!

**Continuity**

\( f: \mathbb{R}^2 \to \mathbb{R} \) is continuous at \((a,b)\) if

\( \lim_{(x,y) \to (a,b)} f(x,y) = f(a,b) \)

\( f: D \to \mathbb{R} \) is continuous on \( D \) if it is continuous at each \((a,b)\) in \( D \)

**Ex.** Every polynomial is continuous everywhere (why?)

**Ex.** Every rational \( f(x,y) = \frac{P(x,y)}{Q(x,y)} \) is continuous everywhere except where \( Q(a,b) = 0 \). (why? \( Q(x,y) \))

**Ex.** Where is \( f(x,y) = \frac{x^2+y^2}{x^2+y^2} \) continuous? Certainly for all \((x,y) \neq (0,0)\).

We also showed previously that \( f \) did not have a limit as \((x,y) \to (0,0)\). So definitely not redefinable as continuous at \((0,0)\).
\[ g(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases} \]
Yes, continuous everywhere! (why?)

Example: Where is \( f(x,y) = \arctan \left( \frac{y}{x} \right) \) continuous?
\( g(x,y) \) is continuous except at \( x=0 \). \( \arctan \) is continuous everywhere. So \( f \) is continuous wherever \( x \neq 0 \).

Remark: That all this generalizes to \( f \) of \( n \) variables!

\[ \text{Sec 14.3} \]
\[ \frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \{ y \text{ is held fixed} \} \]
\[ \frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \quad \{ x \text{ is held fixed} \} \]

Example: \( f(x,y) = x^3 + x^2y^3 - 2y^2 \), find \( f_x(2,1) \), \( f_y(2,1) \)
\[ f_x = 3x^2 + 2xy^3 - 0 \]
\[ f_x(2,1) = 3(2)^2 + 2(2)(1)^3 = 16 \]
\[ f_y = 0 + 3x^2y^2 - 4y \]
\[ f_y(2,1) = 3(2)^2(1)^2 - 4(1) = 8 \]

Now go to pg 904 of the text, and interpret \( f_x \) and \( f_y \) as slope of tangents to traces!

Remark: Some usual rules apply: \( D(\text{sum}) = \text{sum } D \), \( D(kf) = k Df \)
\( D(\text{prod}) = \ldots \), \( D(\text{quot}) = \ldots \), \( D(f \circ g) = f(g) \cdot Df \) (Chain rule)
\[
\text{Ex: } f(x, y) = \sin \left( \frac{x}{1+y} \right)
\]
Here use the chain rule for functions of one var
\[
f_x = \cos \left( \frac{x}{1+y} \right) \frac{2}{\partial x} \left( \frac{x}{1+y} \right) = \frac{1}{1+y} \cos \left( \frac{x}{1+y} \right)
\]
\[
f_y = \cos \left( \frac{x}{1+y} \right) \frac{2}{\partial y} \left( \frac{x}{1+y} \right)
\]
\[
= -\frac{x}{(1+y)^2} \cos \left( \frac{x}{1+y} \right)
\]