Matrix games

A vector \( y \in \mathbb{R}^m \) is stochastic if \( y_i \geq 0 \) for every \( i \in \{1, \ldots, m\} \) and \( \sum_{i=1}^{m} y_i = 1 \). Throughout, assume \( x, y \) are stochastic vectors in \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. Let \( A = \{a_{ij}\} \) be an \( m \times n \) payoff matrix for a game with zero sum. If the first player chooses his/her strategy \( i \) with probability \( y_i \) for every \( i = 1, \ldots, m \), and the second player chooses his/her strategy \( j \) with probability \( x_j \) for all \( j = 1, \ldots, n \) then the expectation of the profit of the first player will be

\[
F(A, y, x) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_i x_j = y^T Ax.
\]

Thus the first player can provide the expected profit \( v_1(A) = \max_y \min_x F(A, y, x) \) and the second player’s expected loss can be made at most \( v_2(A) = \min_x \max_y F(A, y, x) \). It is not hard to see that \( v_1(A) \leq v_2(A) \) for every payoff matrix \( A \).

We need the following lemma.

**Lemma.** For any payoff matrix \( A \) and stochastic \( y \in \mathbb{R}^m \), \( \min_x y^T Ax = \min_j \sum_{i=1}^{m} y_i a_{i,j} \). And for any stochastic \( x \in \mathbb{R}^n \), \( \max_y y^T Ax = \max_i \sum_{j=1}^{n} a_{i,j} x_j \).

**Proof.** Let \( t = \min_j \sum_{i=1}^{m} y_i a_{i,j} \). We have that

\[
y^T Ax = \sum_{j=1}^{n} \sum_{i=1}^{m} y_i a_{i,j} x_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} y_i a_{i,j} \geq \sum_{j=1}^{n} x_j t = t,
\]

so \( \min_x y^T Ax \geq t \). Furthermore, for any \( j \in \{1, \ldots, n\} \),

\[
\min_x y^T Ax \leq y^T A e_j = \sum_{i=1}^{m} y_i a_{i,j},
\]

where \( e_j \in \mathbb{R}^n \) is the \( j \)th standard basis vector. Hence, \( \min_x y^T Ax \leq \min_j y^T A e_j = t \) and the conclusion follows.

The proof of the second sentence is similar. \( \square \)

**Theorem.** For every payoff matrix \( A \), \( v_1(A) = v_2(A) \).

**PROOF.** Consider the following LP1:

\[
\begin{align*}
\text{Find} & \quad \max v_1 \\
such that & \quad \begin{array}{cccccc}
 v_1 & -a_{11}y_1 & -a_{12}y_2 & \ldots & -a_{1m}y_m & \leq & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \ldots \\
 v_1 & -a_{n1}y_1 & -a_{n2}y_2 & \ldots & -a_{nm}y_m & \leq & 0 \\
y_1 & +y_2 & \ldots & +y_m & = & 1 \\
 v_1 & \text{unconstrained} \\
y_i & \geq & 0 & \forall i
\end{array}
\end{align*}
\]

Using the lemma, one can check that the maximum possible \( v_1 \) in this LP is exactly \( v_1(A) \).
Similarly, $v_2(A)$ is the solution of the following LP2:

Find $\min v_2$

\[
\begin{bmatrix}
    x_1 & x_2 & \ldots & x_n \\
    -a_{11}x_1 & -a_{12}x_2 & \ldots & -a_{1n}x_n + v_2 & \geq 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    -a_{m1}x_1 & -a_{m2}x_2 & \ldots & -a_{mn}x_n + v_2 & \geq 0 \\
    x_j & \geq 0 & \forall j \\
v_2 & \text{unconstrained}
\end{bmatrix}
\]

such that

Both these problems have feasible solutions (any pure strategies would do). Moreover, they are dual. This proves the theorem.