# Triangle factors of graphs without large independent sets and of weighted graphs

József Balogh* Theodore Molla† Maryam Sharifzadeh‡

with an Appendix by Christian Reiher§ and Mathias Schacht¶

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**Abstract**

The classical Corrádi-Hajnal theorem claims that every $n$-vertex graph $G$ with $\delta(G) \geq 2n/3$ contains a triangle factor, when $3|n$. In this paper we present two related results that both use the absorbing technique of Rödl, Ruciński and Szemerédi.

Our main result determines the minimum degree condition necessary to guarantee a triangle factor in graphs with sublinear independence number. In particular, we show that if $G$ is an $n$-vertex graph with $\alpha(G) = o(n)$ and $\delta(G) \geq (1/2 + o(1))n$, then $G$ has a triangle factor and this is asymptotically best possible. Furthermore, it is shown for every $r$ that if every linear size vertex set of a graph $G$ spans quadratically many edges, and $\delta(G) \geq (1/2 + o(1))n$, then $G$ has a $K_r$-factor for $n$ sufficiently large. We also propose many related open problems whose solutions could show a relationship with Ramsey-Turán theory.

Additionally, we also consider a fractional variant of the Corrádi-Hajnal Theorem, settling a conjecture of Balogh-Kemkes-Lee-Young. Let $t \in (0,1)$ and $w : E(K_n) \to [0,1]$. We call a triangle *$t$-heavy* if the sum of the weights on its edges is more than $3t$. We prove that if $3|n$ and $w$ is such that for every vertex $v$ the sum of $w(e)$ over edges $e$ incident to $v$ is at least $\left(\frac{1+2t}{3} + o(1)\right)n$, then there are $n/3$ vertex disjoint heavy triangles in $G$.

*Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA jobal@math.uiuc.edu. Research is partially supported by Simons Fellowship, NSA Grant H98230-15-1-0002, NSF Grant DMS-1500121, Arnold O. Beckman Research Award (UIUC Campus Research Board 15006) and Marie Curie Fellowship FP7-PEOPLE-2012-IIF 327763.

†Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA molla@illinois.edu. Research is partially supported by NSF Grant DMS-1500121.

‡Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA sharifz2@illinois.edu.

§Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, D-20146 Hamburg, Germany, Christian.Reiher@uni-hamburg.de.

¶Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, D-20146 Hamburg, Germany, schacht@math.uni-hamburg.de. Research supported by the Heisenberg-Programme of the DFG.

# 1 Introduction

Given an $n$-vertex graph $G$ and an $h$-vertex graph $H$, an *$H$-tiling* is a collection of vertex disjoint copies of $H$ in $G$. A *perfect $H$-tiling* or an *$H$-factor* is an $H$-tiling that covers all of the vertices of $G$. One obvious necessary condition for an $H$-factor in $G$ is $h|n$. Throughout the rest of the paper we will assume that this divisibility condition holds whenever necessary. We also always assume that $n$ is sufficiently large.

For a given graph $H$, a fundamental problem in graph theory is to find sufficient conditions for a graph $G$ to have an $H$-factor. A classical result of Tutte gives necessary and sufficient conditions for the case $H = K_2$. Another celebrated result of this type is the Hajnal-Szemerédi Theorem [15] which states that every $n$-vertex graph $G$ with $\delta(G) \geq (1 - 1/r)n$ has a $K_r$-factor. The case $r = 3$ was proved earlier by Corrádi and Hajnal [8]. The almost balanced complete $r$-partite graph on $n$ vertices shows that the minimum degree condition in the Hajnal-Szemerédi theorem is sharp. This extremal example, which is very similar to the Turán graph, has chromatic number $r$, has an independent set of size greater than $n/r$, it is almost regular and it is very far from random-like.

Although the Hajnal-Szemerédi Theorem was proved many years ago, there has been significant recent activity on related theorems. For example, Alon-Yuster [2], Komlós-Sárközy-Szemerédi [21] and Kühn-Osthus [23] have all proved theorems similar to the Hajnal-Szemerédi Theorem where complete graphs factors are replaced with $H$-factors where $H$ is an arbitrary graph; Kierstead-Kostochka proved the Hajnal-Szemerédi Theorem with an Ore-type degree condition [20]; Fischer [14], Martin-Szemerédi [25], and Keevash-Mycroft [18] have proved multipartite variants; and Wang [30], Keevash-Sudakov [19], Czygrinow-Kierstead-Molla [10], Czygrinow-DeBiasio-Kierstead-Molla [9], Treglown [29] and Balogh-Lo-Molla [5] have all proved analogues of the Hajnal-Szemerédi Theorem in directed and oriented graphs.

Erdős and Sós [13] began studying a variation on Turán's theorem that excludes graphs with high independence number such as Turán graph. They investigated the maximum number of edges in an $n$-vertex, $K_r$-free graph with independence number $o(n)$. These types of problems became known as Ramsey-Turán problems, and have been studied extensively over the past 40 years, see for example [4, 11, 12, 26, 27]. The following question is a Ramsey-Turán type of variant of the Hajnal-Szemerédi theorem.

**Question 1.1.** *Let $G$ be an $n$ vertex graph with $\alpha(G) = o(n)$. What is the minimum degree condition on $G$ that guarantees a $K_k$-factor in $G$ for $k \geq 3$?*

As we mentioned earlier, the main motivation for Question 1.1 is the fact that the extremal example for the Hajnal-Szemerédi theorem is a very structured graph. Krivelevich-Sudakov-Szabó [22] considered a pseudo-random version of the Corrádi-Hajnal theorem, which was later improved and extended by Allen-Böttcher-Hàn-Kohayakawa-Person [1]. In particular, they proved that every $n$-vertex graph $G$ satisfying some pseudo-random conditions has a triangle factor. Their result is relevant when $\delta(G) \gg n^{3/4}$, but as the minimum degree condition is weakened a stronger pseudo-random condition is required. This pseudo-random condition always implies $\alpha(G) = o(n)$. In fact, their pseudo-random condition implies that the graph has uniform edge distribution, a much stronger condition than $\alpha(G) = o(n)$, we recommend reading [1] for the precise statement. In Question 1.1, since we only require $\alpha(G) = o(n)$, we need a much higher minimum degree condition. Our first main result is to answer Question 1.1 for $k = 3$.

**Theorem 1.2.** *For every $\varepsilon > 0$, there exists $\gamma > 0$ and $n_0$ such that the following holds. For every $n$-vertex graph $G$ with $n > n_0$, if $\delta(G) \geq (1/2 + \varepsilon)n$ and $\alpha(G) \leq \gamma n$, then $G$ has a triangle factor.*

The following examples show that the minimum degree condition in the statement of Theorem 1.2 is asymptotically best possible. For $n = 2k$, consider the graph $G_1 = K_{k-1} \cup K_{k+1}$. This graph does not have a triangle factor and $\delta(G_1) = n/2 - 2$. Another example for $n = 2k$ is the following. Consider the graph $G_2$ consisting of $K_{k+2}$ and $K_{k-1}$ sharing one vertex. Since $3|2k$, we have that both $k + 2 \equiv 2 \pmod 3$ and $k - 1 \equiv 2 \pmod 3$. Hence, this graph has no triangle factor and $\delta(G_2) = n/2 - 2$. For $n = 2k + 1$ consider the graph $G_3$ consisting of two copies of $K_{k+1}$ sharing one vertex. Since $3|2k + 1$, we have $k + 1 \equiv 2 \pmod 3$. Hence, this graph has no triangle factor and $\delta(G_3) = (n - 1)/2$.

In the Appendix it is shown for every $r$ that if every linear size vertex set of a graph $G$ spans quadratically many edges, and $\delta(G) \geq (1/2 + o(1))n$, then $G$ has a $K_r$-factor for $n$ sufficiently large.

We also prove the triangle case of the conjecture proposed by Balogh-Kemkes-Lee-Young ([3], Conjecture 1). Let $t \in (0, 1)$ and $w : E(K_n) \to [0, 1]$. We call $x, y, z \in V(K_n)$ a *$t$-heavy triangle* if $w(xy) + w(xz) + w(yz) > 3t$, for $v \in V(K_n)$ we write $d_w(v)$ for the sum of the weights on the edges incident to $v$ and let $\delta_w(K_n) = \min_{v \in V(K_n)} d_w(v)$.

**Theorem 1.3.** *For any $t \in (0, 1)$ and $\varepsilon > 0$ there exists $n_0$ such that for $3k = n \geq n_0$, if $w : V(K_n) \to [0, 1]$ is such that $\delta_w(K_n) \geq \left(\frac{1+2t}{3} + \varepsilon\right) n$ then there are $k$ vertex-disjoint $t$-heavy triangles in $K_n$.*

This theorem is asymptotically best possible for every $t \in (0, 1)$ by the following example from [3]. Let $n$ be divisible by 3, and let $U \subseteq V(K_n)$ be such that $|U| = 2n/3 + 1$ and, for every $e \in E(K_n)$, set $w(e) = t$ if $e$ has both endpoints in $U$ and otherwise set $w(e) = 1$. Since every $t$-heavy triangle intersects $U$ in at most two vertices, there are no $n/3$ vertex disjoint $t$-heavy triangles. Furthermore, we have that $\delta_w(K_n) = n - |U| + t(|U| - 1) = (1 + 2t)n/3 - 1$.

As was pointed out in [3], when $t = 2/3$ and $w(e) \in \{0, 1\}$ for every $e \in E(K_n)$, the Corrádi-Hajnal Theorem implies the existence of a $t$-heavy triangle factor when $\delta_w(K_n) \geq 2n/3$. It is interesting to note that when $w(e)$ is allowed to take any value in $[0, 1]$ we can show that we must force $\delta_w(K_n)$ to be greater than $7n/9 - 1$ to guarantee a $t$-heavy triangle factor by replacing $t$ with $2/3$ in the example above.

**Notation.** Most of the notation that we use is standard. If a set has cardinality $k$, then we say that it is a $k$-set. For a collection $\mathcal{U}$ of subsets of $V(G)$ we let $V(\mathcal{U}) := \bigcup_{U \in \mathcal{U}} U$. Similarly, for a collection $\mathcal{U}$ of subgraphs of $G$ we let $V(\mathcal{U}) := \bigcup_{U \in \mathcal{U}} V(U)$. For any $v \in V$

and $U \subseteq V$, we let $d_U(v) = d(v, U)$ be the number of edges incident to $v$ and a vertex in $U$. For $U, W \subseteq V$, we let $e_G(U, W) := \sum_{u \in U} d(u, W)$.

We use the notion of a multiset in several places, and when $U$ is a multiset, we write $\nu_U(u)$ to represent the multiplicity of the element $u \in U$.

The notation $a \ll b$ means that there exists an increasing function $f$ such that when $a$ and $b$ are constants and $a \leq f(b)$ the argument holds. The function $f$ is not always explicitly specified, but could be computed.

**Outline of the paper.** In Section 2, we state the two main lemmas of Theorem 1.2 and the two main lemmas of Theorem 1.3 and show how they are used to prove the theorems. In Section 3 we introduce and prove tools for the absorbing method. In Section 4, we prove the two main lemmas of Theorem 1.2. In Section 5, we prove the main lemmas of Theorem 1.3. The Appendix contains the proof of a result on the existence of $K_r$-factors in graphs.

# 2 Main Lemmas and the proof of the theorems

The absorbing method of Rödl, Ruciński and Szemerédi [28] is used in both the proof of Theorem 1.2 and the proof of Theorem 1.3. Following the absorbing method, the heart of the proof of Theorem 1.2 is the following two lemmas.

**Lemma 2.1** (Absorbing Lemma for Theorem 1.2)**.** *For $0 < \gamma \ll \zeta \ll \sigma \ll \varepsilon < 1/6$ the following holds. If $G = (V, E)$ is a graph such that $\delta(G) \geq (1/2 + \varepsilon)n$ and $\alpha(G) \leq \gamma n$, then there exists $U \subseteq V$ such that $|U| \leq 2\sigma n$ and for every $W \subseteq V \setminus U$ such that $|W|$ is at most $\zeta n$ and divisible by 3, $G[U \cup W]$ has a triangle factor.*

**Lemma 2.2** (Triangle Covering Lemma for Theorem 1.2)**.** *For every $\varepsilon > 0$, there exists $\gamma > 0$ and $n_0$ such that the following holds. For every $n$-vertex graph $G$ with $n > n_0$, $\delta(G) \geq (1/2 + \varepsilon)n$ and $\alpha(G) \leq \gamma n$, there is a triangle tiling on all but at most $16/\varepsilon + 1$ vertices.*

We now show how these two lemmas imply the theorem.

*Proof of Theorem 1.2.* Let $0 < \gamma \ll \zeta \ll \sigma \ll \varepsilon < 1/6$ be as in Lemma 2.1 and such that $\gamma$ is small enough so that Lemma 2.2 holds when $\varepsilon$ and $\gamma$ are replaced with $\varepsilon' := \varepsilon - 2\sigma$ and $\gamma' := \gamma/(1 - 2\sigma)$, respectively. Let $U \subseteq V$ be a set of size at most $2\sigma n$ that is guaranteed by Lemma 2.1 and let $V' := V \setminus U$, $n' := |V'|$ and $G' := G[V']$. Note that $\delta(G') \geq (1/2 + \varepsilon')n'$ and $\alpha(G') \leq \gamma n \leq \gamma' n'$, so Lemma 2.2 implies that there exists a triangle tiling $\mathcal{T}_1$ such that if $W := V' \setminus V(\mathcal{T}_1)$, then $|W| \leq 16/\varepsilon' + 1$. Since $n$ is divisible by 3, $|W|$ is divisible by 3 and Lemma 2.1 implies that there exists a triangle factor $\mathcal{T}_2$ of $G[W \cup U]$, and $\mathcal{T}_1 \cup \mathcal{T}_2$ is a triangle factor of $G$. □

We prove Theorem 1.3 in roughly the same way. That is, we prove the following absorbing lemma (Lemma 2.3) and almost tiling lemma (Lemma 2.4) and then we use them both to obtain the desired result. We omit the details of proving Theorem 1.3, given Lemma 2.3 and Lemma 2.4, since they are identical to the details in the proof of Theorem 1.2.

**Lemma 2.3** (Absorbing Lemma for Theorem 1.3)**.** *For every $t \in (0,1)$ let $0 < \zeta \ll \sigma \ll \varepsilon < 1$ and $n_0$ such that the following holds. For every $n \geq n_0$ that is divisible by 3, graph $G = (V, E) = K_n$ and $w : E \to [0,1]$ such that $\delta_w(G) \geq \left(\frac{1+2t}{3} + \varepsilon\right) n$, there exists $U \subset V$ such that $|U| \leq \sigma n$ and for any $W \subseteq V \setminus U$ such that $|W|$ is at most $\zeta n$ and divisible by 3, there exists a perfect tiling of $G[U \cup W]$ with $t$-heavy triangles.*

**Lemma 2.4** (Triangle Covering Lemma for Theorem 1.3)**.** *For every $\varepsilon > 0$ there exists $n_0$ such that for every $n \geq n_0$, if $G = (V, E) = K_n$ and $w : E \to [0,1]$ such that $\delta_w(G) \geq \left(\frac{1+2t}{3} + \varepsilon\right) n$ then there is a $t$-heavy triangle tiling on all but at most 6 vertices.*

The proof of the absorbing lemma for Theorem 1.3 (Lemma 2.3), while non-trivial, is standard within the context of the absorbing method. However, the absorbing lemma for Theorem 1.2 (Lemma 2.1) is more involved. The framework for the proof of Lemma 2.1 is established in Section 3. This framework will also be used in the proof of Lemma 2.3, but most of it is not necessary for Theorem 1.3.

In most applications of the absorbing method, for say triangle-factors, it is proved that there exists a constant $k$ such that for every 3-set $X \subseteq V(G)$, there are $\Omega(n^{3k})$ $3k$-sets $U \subseteq V(G)$ such that both $G[U]$ and $G[X \cup U]$ have triangle factors. In our setting, this is not necessarily true as the following example illustrates.

**Example 2.5.** Fix $k \in \mathbb{N}$ and $0 < \varepsilon < 1/6$. Let $V_1, V_2, \ldots, V_{2m+1}$ be disjoint sets that partition $[n]$ where $|V_1| = \lfloor (1/2 - \varepsilon)n \rfloor$ and $|V_2|, \ldots, |V_{2m+1}| \geq \lceil 2\varepsilon n \rceil$. Note that $m$ can be as large as $\left\lfloor \frac{\lceil n/2+\varepsilon n \rceil}{2\lceil 2\varepsilon n \rceil} \right\rfloor \geq \varepsilon^{-1}/8$. Let $G'$ be the graph on $[n]$, where for every $i \in [m]$ we add all possible edges between $V_1$, $V_{2i}$, $V_{2i+1}$, i.e. $G'[V_1 \cup V_{2i} \cup V_{2i+1}]$ is the complete 3-partite graph with parts $V_1$, $V_{2i}$ and $V_{2i+1}$ for every $i \in [m]$. Note that $\delta(G') \geq (1/2 + \varepsilon)n$, and every triangle in $G'$ has exactly one vertex in $V_1$. We form $G$ by, for every $i \in [2m+1]$, adding edges with both endpoints in $V_i$ so that so that $d_G(v, V_i) = o(n)$ for every $v \in V_i$ and $\alpha(G[V_i]) = o(n)$. It is well-known that this is possible. Let $G'' := G - G'$.

**Claim 2.6.** *For every fixed $k$, there exists a 3-set $X \subseteq V$ such that there are only $o(n^{3k})$ $3k$-sets $U \subseteq V$ such that both $G[U]$ and $G[U \cup X]$ have a triangle factor.*

*Proof.* Let $\{x_1, x_2, x_3\} := X \subseteq V \setminus V_1$ such that $X$ is an independent set and $|X \cap (V_{2i} \cup V_{2i+1})| \neq 3$ for every $i \in [m]$. Let $U \subseteq V$ such that $G[U]$ has a triangle factor $\mathcal{T}_1$ and $G[U \cup X]$ has a triangle factor $\mathcal{T}_2$. If $E(G''[U \cup X]) = \emptyset$, then every $T \in \mathcal{T}_1 \cup \mathcal{T}_2$ has exactly one vertex in $V_1$, so $|U \cap V_1| = k$ and $|(X \cup U) \cap V_1| = k + 1$, but this contradicts the fact that $X \cap V_1 = \emptyset$. Therefore, $E(G''[U \cup X]) \neq \emptyset$, but there are only $o(n^{3k})$ $3k$-sets $U \subseteq V$ such that $G''[U \cup X]$ contains an edge. □

To get around this problem, for a fixed constant $k$, we construct an absorber $A \subseteq V$ by a standard probabilistic argument. Typically, with the absorbing method, it is shown if $\mathcal{X}$ is any small collection of pairwise disjoint 3-sets of vertices, then $G[A \cup V(\mathcal{X})]$ has a triangle factor. The difference, compared to standard absorbers, is that here we only show that $G[A \cup V(\mathcal{X})]$ has a triangle factor when, for every $X \in \mathcal{X}$, there exist $\Omega(n^{3k})$ $3k$-sets $U$

such that both $G[U]$ and $G[X \cup U]$ have triangle factors. The absorbing structure consists of the absorber $A$ and an absorbable family of sets $\mathcal{N}$ together with a partition of $\mathcal{N}$ into disjoints families $\mathcal{N}_1, \dots, \mathcal{N}_\ell$ such that, for every $i \in [\ell]$, $|\mathcal{N}_i| = 3$ and if we form a 3-set of vertices $X$ by taking one vertex from each of the three sets in $\mathcal{N}_i$, there are $\Omega(n^{3k})$ $3k$-sets $U$ such that both $G[U]$ and $G[X \cup U]$ have triangle factors, i.e. the set $X$ can be part of a collection absorbed by $A$. We also ensure that every vertex in the graph has a large number of neighbors into $V(\mathcal{N})$. The set $U := A \cup V(\mathcal{N})$ is then the absorber from the statement of Lemma 2.1. Indeed, given a set $W$ in $V \setminus U$ to absorb, we start constructing a triangle factor of $G[U \cup W]$ by greedily finding a set of vertex disjoint triangles such that every vertex in $W$ is in exactly one triangle. This is possible because $\alpha(G)$ and $|W|$ are both small relative to the number of neighbors every vertex in $W$ has in the set $V(\mathcal{N})$. We then find vertex disjoint triangles inside $V(\mathcal{N})$ so that, when we remove these triangles, the three sets contained in each $\mathcal{N}_i$ have the same size. This makes it possible to partition the remaining vertices in $V(\mathcal{N})$ into 3-sets such that each can be absorber by $A$. Therefore, we have a triangle factor in the graph induced by the union of the remaining vertices in $V(\mathcal{N})$ and $A$.

# 3 Tools for the absorbing method

In this section, we prove lemmas that are used in both the proof of Theorem 1.2 and Theorem 1.3. When reading this section in the context of Theorem 1.3, all references to triangles should be interpreted as references to $t$-heavy triangles.

We will refer to the following theorem throughout as the Chernoff bound, see e.g. Corollary 2.3, Theorem 2.8 and Theorem 2.10 in [17].

**Theorem 3.1.** *Let $X$ be a hypergeometric random variable or let $X = \sum_{i=1}^{n} X_i$ where $X_1, \dots, X_n$ are independent random indicator variables. If $0 < \lambda \leq 3/2$, then*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \lambda \mathbb{E}X) \leq 2 \exp\left(-\frac{\lambda^2}{3}\mathbb{E}X\right). \tag{1}$$

*In particular,* (1) *applies when $X$ is a binomial random variable.*

**Definition 3.2.** Let $G(V, E)$ be an $n$-vertex graph. Distinct vertices $x, y \in V$ are $(c, k)$-*linked* if there are at least $(cn)^{3k-1}$ multisets $U \subseteq V$ of size $3k - 1$ such that the following holds. Let $U'$ be the set of elements of $U$, without repetition. Then, both $G[U' \cup \{x\}]$ and $G[U' \cup \{y\}]$ have triangle factors in the following sense: if a vertex in $U$ has multiplicity $i$ then it should be in exactly $i$ triangles. We also call $U$ a $k$-*linking set* for $\{x, y\}$.

For a vertex $v \in V$, denote by $L_{c,k}(v)$ the set of vertices that are $(c, k)$-linked with $v$. A set $V' \subseteq V$ is $(c, k)$-*linked* if every pair of vertices in $V'$ are $(c, k)$-linked.

Informally speaking, the goal of this section is to provide the tools with which we can construct the pairwise disjoint vertex sets $\mathcal{N}$ and the partition $\{\mathcal{N}_1, \dots, \mathcal{N}_\ell\}$ of $\mathcal{N}$ referenced in the discussion at the end of Section 2 (see Definition 3.7). That is, we want each part $\mathcal{N}_i$ to consist of three sets with the property that if we form a 3-set $X$ by taking one vertex from each of the three sets in $\mathcal{N}_i$, there are $\Omega(n^{3k})$ $3k$-sets $U$ such that both $G[U]$ and $G[X \cup U]$ have triangle factors. Additionally, we want $V(\mathcal{N})$ to be small, but with the property that every vertex in the graph has a significant number of neighbors in $V(\mathcal{N})$. Now suppose that we have a partition of $\{V_1, \dots, V_d\}$ of $V(G)$ such that each part is $(c,k)$-linked for some constants $c$ and $k$ (see Definition 3.3), and that for some $V_i$,$V_j$ and $V_k$ there are $\Omega(n^3)$ triangles with one vertex in each of $V_i$, $V_j$ and $V_k$ (see Definition 3.6). Then it is not hard see that if we form a 3-set $X$ by taking one vertex from each of $V_i$, $V_j$ and $V_k$, there are $\Omega(n^{3k})$ $3k$-sets $U$ such that both $G[U]$ and $G[X \cup U]$ have triangle factors. Therefore, using the Chernoff and union bounds, we could attempt to form our desired collection $\mathcal{N}$ by taking small, randomly selected subsets from every $V_i$, $V_j$ and $V_k$ for which there are $\Omega(n^3)$ triangles with one vertex in each subset. This is essentially how we form $\mathcal{N}$ in our proof of Lemma 2.1. In what follows, we do not require the indices $i, j$ and $k$ to be distinct, so it is convenient for us to use the notion of multisets.

**Definition 3.3.** For $k \in \mathbb{N}$ and $0 < \phi < \psi \leq 1$, call a partition $\mathcal{M} = \{V_1, \dots, V_d\}$ of $V$ $(\psi, \phi, k)$-*linked* if $|V_i| \geq \psi n$ and $V_i$ is $(\phi, k)$-linked for every $i \in [d]$. Note that $d \leq 1/\psi$.

In Example 2.5, for every $i \in [2m+1]$, $V_i$ is $(\varepsilon, 1)$-linked, in particular, $\{V_1, \dots, V_{2m+1}\}$ is a $(2\varepsilon, \varepsilon, 1)$-linked partition of $G$.

**Claim 3.4.** *Consider the graph from Example 2.5. For every $k \in \mathbb{N}$ and $\phi > 0$, if $v_i \in V_i$ and $v_j \in V_j$ where $i \neq j$, then $v_i$ and $v_j$ are not $(\phi, k)$-linked.*

*Proof.* We will show that there are only $o(n^{3k-1})$ $k$-linking multisets for $\{v_i, v_j\}$. Let $U$ be such a multiset. Since there are only $o(n^{3k-1})$ multisets of order $3k-1$ such that an element of $U$ has multiplicity greater than 1, we can assume that $U$ is actually a set. Furthermore, we can assume that both $G''[U+v_i]$ and $G''[U+v_j]$ are independent sets, since there are only $o(n^{3k-1})$ sets of order $3k-1$ that do not have this property. This implies that both $U+v_i$ and $U+v_j$ have exactly $k$ vertices in $V_1$, so, since $i \neq j$, neither $i$ nor $j$ is 1. Therefore, we can assume without loss of generality that $i$ is even. Hence, $|(U+v_i) \cap V_i| = |(U+v_i) \cap V_{i+1}|$ and $|(U+v_j) \cap V_i| = |(U+v_j) \cap V_{i+1}|$, which is impossible since $i \neq j$. □

Now we study properties of a linked partition of a graph.

**Proposition 3.5.** *For a graph $G = (V, E)$, let $x_1, x_2 \in V$, $k_1, k_2 \in \mathbb{N}$, $c, c_1, c_2 > 0$, $k := k_1 + k_2$ and $c' := \min\{c, c_1, c_2\}$. If*

$$|L_{c_1,k_1}(x_1) \cap L_{c_2,k_2}(x_2)| \geq cn,$$

*then $x_1$ and $x_2$ are $(\frac{1}{3}c', k)$-linked.*

*Proof.* Assume $k_1 \leq k_2$. Let $(x, U_1, U_2)$ be an ordered triple such that $x \in L_{c_1,k_1}(x_1) \cap L_{c_2,k_2}(x_2)$ and $U_i$ is a $k_i$-linking set for $\{x_i, x\}$ and $i \in [2]$. There are at least

$$cn \cdot (c_1 n)^{3k_1-1} \cdot (c_2 n)^{3k_2-1} \geq (c' n)^{3k_1+3k_2-1}$$

such ordered triples and if $U := \{x\} \cup U_1 \cup U_2$ then $U$ is a $(k_1 + k_2)$-linking set for $\{x_1, x_2\}$. Let $(x', U_1', U_2')$ be another such triple such that $U = \{x'\} \cup U_1' \cup U_2'$. By first picking $x'$ and then $U_1'$ from the multiset $U$ (and using the fact that $x + 1 \leq 3 \cdot (3/2)^x$ for $x > 0$), we have that there at most

$$(3k_1 + 3k_2 - 1) \cdot \binom{3k_1 + 3k_2 - 2}{3k_1 - 1} \leq \left(3 \cdot \left(\frac{3}{2}\right)^{3k_1+3k_2-2}\right) \cdot 2^{3k_1+3k_2-2} = 3^{3k_1+3k_2-1}$$

such triples $(x', U_1', U_2')$ and the conclusion follows. □

**Definition 3.6.** Given a partition $\mathcal{M} = \{V_1, \ldots, V_d\}$ of $V$, $0 < \phi < 1$ and an arbitrary multiset $I$ of $[d]$ of order 3, let $t(\mathcal{M}, I)$ be the number of triangles $T$ such that $|V(T) \cap V_i| = \nu_I(i)$ (recall that $\nu_I(i)$ is the multiplicity of $i$ in the multiset $I$). For every $1 \leq i \leq d$ and let

$$F_\phi(\mathcal{M}) = \{I : t(\mathcal{M}, I) > \phi n^3\}. \tag{2}$$

Also, for $i \in [d]$, let $t_\phi(\mathcal{M}, i)$ be the number of times the index $i$ appears in a multiset of $F_\phi(\mathcal{M})$ with multiplicity, i.e. $3 \cdot |F_\phi(\mathcal{M})| = \sum_{i=1}^{d} t_\phi(\mathcal{M}, i)$. When the partition $\mathcal{M}$ is clear from context, we often use $F_\phi$ and $t_\phi(i)$ to refer to $F_\phi(\mathcal{M})$ and $t_\phi(\mathcal{M}, i)$, respectively. For convenience, we let $k : F_\phi(\mathcal{M}) \times [3] \to [d]$ be the map defined by $\{k(I, 1), k(I, 2), k(I, 3)\} = I$, where $k(I, 1) \leq k(I, 2) \leq k(I, 3)$ for every $I \in F_\phi(\mathcal{M})$.

For the graph from Example 2.5, $F_{\varepsilon^2}(\{V_1, \ldots, V_{2m+1}\}) = \{\{1, 2i, 2i + 1\} : i \in [m]\}$, $k(\{1, 2i, 2i + 1\}, 1) = 1$, $k(\{1, 2i, 2i + 1\}, 2) = 2i$, and $k(\{1, 2i, 2i + 1\}, 3) = 2i + 1$ for every $i \in [m]$.

**Definition 3.7.** Given constants $0 < \eta < \phi < \psi \leq 1$ and a partition $\mathcal{M} = \{V_1, \ldots, V_d\}$ of $V$ and $A \subseteq V$, a collection $\mathcal{N}$ of vertex disjoint subsets of $V \setminus A$ is called *$(\mathcal{M}, \phi, \eta)$-absorbable (with respect to $A$)* if there exists a bijective map $X : F_\phi(\mathcal{M}) \times [3] \to \mathcal{N}$ such that for all $I \in F_\phi(\mathcal{M})$

- $X(I, j) \subseteq V_{k(I,j)}$ for every $j \in [3]$ and
- $|X(I, 1)| = |X(I, 2)| = |X(I, 3)| \leq \eta n$.

For every $(\mathcal{M}, \phi, \eta)$-absorbable collection $\mathcal{N}$ we will always implicitly assume that a fixed function $X$ exists. Call $A$ an *$(\mathcal{M}, \phi, \eta)$-absorber* if for any collection $\mathcal{N}$ that is $(\mathcal{M}, \phi, \eta)$-absorbable with respect to $A$, $G[A \cup V(\mathcal{N})]$ has a triangle factor.

For $d = 1$, Lemma 3.8 is very similar to lemmas that appear in other results which use the absorbing method and the proof is nearly identical. For example, see Lemma 1.1 in [24] for a general result that is used to find minimum degree conditions for the existence of hypergraph matchings.

**Lemma 3.8.** *For every $k$ and $0 < \eta \ll \sigma \ll \phi \ll \psi \leq 1$, the following holds. If $G = (V, E)$ is a graph and $\mathcal{M} = \{V_1, \ldots, V_d\}$ is a $(\psi, \phi, k)$-linked partition of $V$, then there exists an $(\mathcal{M}, \phi, \eta)$-absorber $A \subseteq V$ such that $|A| \leq \sigma n$.*

*Proof.* Let $\ell := 9 \cdot k$ and $0 < \eta \ll \xi \ll \sigma$. For a 3-set $W = \{w_1, w_2, w_3\} \subseteq V$ denote by $\mathcal{L}_W$ the set of ordered $\ell$-tuples $(u_1, \dots, u_\ell) \in V^\ell$ such that $u_{3k}u_{6k}u_{9k}$ is a triangle and, for $j \in [3]$, the multiset $\{u_{3k\cdot(j-1)+1}, \dots, u_{3k\cdot j-1}\}$ is a $k$-linking multiset for $\{w_j, u_{3k\cdot j}\}$. Note that if the vertices $u_1, \dots, u_\ell$ are distinct and $U := \{u_1, \dots, u_\ell\}$, then both $G[U]$ and $G[U \cup W]$ have triangle factors. We say that the 3-set $W$ is *acceptable* if $|\mathcal{L}_w| \geq 4(\phi n)^\ell$.

Form a random subset of $\ell$-tuples $\mathcal{A}' \subseteq V^\ell$ where each $\ell$-tuple is picked independently at random with probability $p := \xi n^{1-\ell}$. We have the following:

$$\mathbb{E}|\mathcal{A}'| = p \cdot |V^\ell| = \xi n, \tag{3}$$

$$\mathbb{E}|\mathcal{A}' \cap \mathcal{L}_W| \geq p \cdot 4(\phi n)^\ell = 4\xi\phi^\ell n \text{ for every acceptable 3-set } W. \tag{4}$$

We call a pair of $\ell$-tuples $(u_1, \dots, u_\ell)$ and $(u'_1, \dots, u'_\ell)$ a *bad pair* if a vertex appears more than once in the list $u_1, \dots, u_\ell, u'_1, \dots, u'_\ell$. The number of bad pairs is at most $(2\ell)^2 \cdot n^{2\ell-1}$. Hence,

$$\mathbb{E}|\{\text{bad pairs in } \mathcal{A}'\}| \leq p^2(2\ell)^2 \cdot n^{2\ell-1} = 4\xi^2(\ell)^2 n. \tag{5}$$

Therefore, by Markov's inequality, with probability at least 1/2,

(a) $\mathcal{A}'$ has at most $8\xi^2(\ell)^2 n$ bad-pairs.

Furthermore, since there are at most $\binom{n}{3}$ acceptable sets $W$, the Chernoff bound and the union bound with (3) and (4) imply that w.h.p. $\mathcal{A}'$ is such that

(b) $|\mathcal{A}'| \leq 2\xi n$ and

(c) $|\mathcal{A}' \cap \mathcal{L}_W| \geq 2\xi\phi^\ell n$ for all acceptable 3-sets $W$.

Therefore there exists $\mathcal{A}'$ that satisfies properties (a), (b) and (c). We now remove both elements from every bad pair in $\mathcal{A}'$. We also remove any tuples in $\mathcal{A}'$ that are not in $\mathcal{L}_W$ for any acceptable 3-set $W$. We call $\mathcal{A}$ the remaining part of $\mathcal{A}'$. Note that for every $(u_1, \dots, u_\ell) \in \mathcal{A}$, there is a triangle factor in $G[\{u_1, \dots, u_\ell\}]$. Since $\phi^\ell \geq 16\ell^2\xi$,

$$|\mathcal{A} \cap \mathcal{L}_W| \geq \xi\phi^\ell n \text{ for every acceptable 3-set } W. \tag{6}$$

Let $A$ be the union of the vertices in the $\ell$-tuples of $\mathcal{A}$. We have that $|A| \leq 2\ell\xi n \leq \sigma n$. Let $\mathcal{N}$ be a collection of subsets of $V \setminus A$ that is $(\mathcal{M}, \phi, \eta)$-absorbable with respect to $A$. For every $I \in F_\phi$, $|X(I,1)| = |X(I,2)| = |X(I,3)| \leq \eta n$, so there exists a partition $\mathcal{W}$ of $V(\mathcal{N})$ into parts of size 3 such that for every $W \in \mathcal{W}$ there exists $I \in F_\phi$ such that $W$ has one vertex in each of $X(I,1), X(I,2)$ and $X(I,3)$. Note that

$$|\mathcal{W}| = |V(\mathcal{N})|/3 \leq 3\eta n \cdot |F_\phi|/3 \leq \eta n d^3 \leq \xi\phi^\ell n. \tag{7}$$

We claim that every $W \in \mathcal{W}$ is acceptable. By construction, there exists an $I \in F_\phi$ such that $W$ has one vertex in each of $X(I,1), X(I,2)$ and $X(I,3)$. We can label $W$ as $\{w_1, w_2, w_3\}$ so that $w_j \in X(I,j) \subseteq V_{k(I,j)}$ for each $j \in [3]$. Since $I \in F_\phi(\mathcal{M})$, there are $\phi n^3$ triangles $u_{3k}u_{6k}u_{9k}$ such that $u_{3k\cdot j} \in V_{k(I,j)}$ for $j \in [3]$. Furthermore, for any $j \in [3]$, since $V_{k(I,j)}$ is $(\phi, k)$-linked, there are at least $(\phi n)^{3k-1}$ $k$-linking multisets for $\{w_j, u_{3k\cdot j}\}$, for each $j \in [3]$. Therefore,

$$|\mathcal{L}_W| \geq (\phi n)^{3(3k-1)}\phi n^3 \geq 4(\phi n)^\ell,$$

so $W$ is acceptable.

Hence, by (6) and (7), we can match every $W \in \mathcal{W}$ to a different $\ell$-tuple in $\mathcal{A} \cap \mathcal{L}_W$ to construct a triangle factor of $G[V(\mathcal{N}) \cup A]$. $\square$

# 4 Proof of Theorem 1.2

## 4.1 Proof of Lemma 2.1

First we prove a series of lemmas and claims as preparation for the proof of Lemma 2.1. The first lemma, Lemma 4.1, uses the method of dependent random choice. We use Lemma 4.1 when we construct the linked partition $\mathcal{M}$ in Lemma 4.3.

**Lemma 4.1.** *Let $F$ be a bipartite graph with classes $(A, B)$ and $0 < \varepsilon \leq 1$ be such that $d_F(a) \geq \varepsilon|B|$ for every $a \in A$ and $d_F(b) \geq \varepsilon|A|$ for every $b \in B$. If $B$ is sufficiently large, then for every $0 < \psi < \varepsilon^4/64$ there exists a positive integer $d$ and a collection of disjoint subsets $\{S_1, \dots, S_d\}$ of $B$ such that*

1. *for every $i \in [d]$, $|S_i| \geq \psi|B|$,*
2. *$\left|\bigcup_{i=1}^d S_i\right| \geq (1-\psi)|B|$, and*
3. *for every $i \in [d]$, there are at most $\psi^3|B|^2$ pairs in $b, b' \in S_i$ such that $|N_F(b) \cap N_F(b')| < \psi^4|A|$.*

*Proof.* Since $0 < \varepsilon \leq 1$ and $0 < \psi < \varepsilon^4/64$, we have the following:

$$-\log(\psi/2)/\varepsilon < \left(4\psi^{-1/2}/\varepsilon\right) - 1 = 8\psi^{1/2}/(2\psi \cdot \varepsilon) - 1 < \varepsilon/(2\psi) - 1.$$

Hence, we can pick a positive integer $d$ so that

$$-\log(\psi/2)/\varepsilon < d < \varepsilon/(2\psi). \tag{8}$$

Call a pair $(b, b') \in B^2$ *bad* if $|N_F(b) \cap N_F(b')| < \psi^4|A|$ and let $Z \subseteq B^2$ be the set of bad pairs. Let $U = \{a_1, \dots, a_d\} \subseteq A$ be a set of $d$ vertices selected uniformly at random and independently with repetition from $A$, and define $f_i$ to be the random variable counting $|N_F(a_i)^2 \cap Z|$ for every $i \in [d]$. By (8),

$$\mathbb{E}f_i = \sum_{(b,b') \in Z} \mathbb{P}(a_i \in N_F(b) \cap N_F(b')) = \sum_{(b,b') \in Z} \frac{|N_F(b) \cap N_F(b')|}{|A|} < \sum_{(b,b') \in Z} \psi^4 < \frac{\psi^3}{2d}|B|^2. \tag{9}$$

Let $Y := \{b \in B : b \notin \bigcup_{i=1}^d N_F(a_i)\}$. Using (8), we have that

$$\mathbb{E}|Y| = \sum_{b \in B} \mathbb{P}(N_F(b) \cap U = \emptyset) = \sum_{b \in B} \left(1 - \frac{|N_F(b)|}{|A|}\right)^d \leq (1-\varepsilon)^d |B| \leq e^{-\varepsilon d}|B| < \frac{\psi|B|}{2}.$$

Markov's inequality and the union bound implies that there exist a choice of $\{a_1, \dots, a_d\} \subseteq A$ such that $|N_F(a_i)^2 \cap Z| \le \psi^3 |B|^2$ for every $i \in [d]$, and $|B \setminus \bigcup_{i=1}^d N_F(a_i)| \le \psi|B|$. Fix such an $\{a_1, \dots, a_d\}$ and let $S'_i := N_F(a_i)$ for $i \in [d]$.

To make the sets $S'_i$ disjoint, we use the following probabilistic argument. For every vertex $v \in \bigcup_{i=1}^d S'_i$ we select uniformly at random and independently of other vertices an index $j$ from the set $\{j \in [d] : v \in S'_j\}$, and then assign $v$ to the set $S_j$. At the end of this process, the sets $\{S_1, \dots, S_d\}$ are disjoint, and using (8) we have,

$$\mathbb{E}|S_i| = \sum_{v \in S'_i} |\{j \in [d] : v \in S'_j\}|^{-1} \ge \frac{d_F(a_i)}{d} \ge \frac{\varepsilon |B|}{d} \ge 2\psi|B| \qquad \text{for all } i \in [d].$$

Because each $S_i$ is the sum of independent random indicator variables, the Chernoff bound implies that

$$\mathbb{P}(|S_i| \le \psi|B|) \le 2\exp(-((1/2)^2 \cdot 2\psi|B|)/3) < 1/d \qquad \text{for all } i \in [d],$$

and, with the union bound, there is an assignment such that $|S_i| \ge \psi|B|$ for every $i \in [d]$. □

Recall that $L_{c,k}(v)$ is the set of vertices $u$ for which there are at least $(cn)^{k-1}$ $k$-linking sets for $\{v, u\}$ (see Definition 3.2).

**Proposition 4.2.** *For any $0 < \varepsilon < 1/6$, if $G = (V, E)$ is a graph on $n$ vertices such that $\delta(G) \ge (1/2 + \varepsilon)n$, then for every vertex $v \in V$, $|L_{\varepsilon^2,1}(v)| \ge \frac{3}{2}\varepsilon^2 n$, for $n$ sufficiently large.*

*Proof.* For a vertex $v \in V$ define

$$F(v) := \{(u, e) \in (V - v) \times E : ve \text{ and } ue \text{ are triangles}\}.$$

Since $\delta(G) \ge (1/2 + \varepsilon)n$, we have $e(G[N(v)]) \ge ((1/2 + \varepsilon)n \cdot 2\varepsilon n)/2$ for $n$ sufficiently large, furthermore for every edge $uu' \in E(G[N(v)])$, $|N(u) \cap N(u') - v| \ge 2\varepsilon n - 1$. Hence,

$$|F(v)| \ge \left(\frac{1}{2} + \varepsilon\right) n \cdot \varepsilon n \cdot (2\varepsilon n - 1) \ge \varepsilon^2 n^3. \tag{10}$$

On the other hand,

$$|F(v)| \le (n - |L_{\varepsilon^2,1}(v)|) \cdot (\varepsilon^2 n)^2 + |L_{\varepsilon^2,1}(v)||E| \le \varepsilon^4 n^3 + |L_{\varepsilon^2,1}(v)| \frac{n^2}{2}. \tag{11}$$

Since $\varepsilon < 1/6$, (10) and (11) imply that $|L_{\varepsilon^2,1}(v)| \ge \frac{3}{2}\varepsilon^2 n$. □

For reference, we now list the relationship between the constants used in the rest of this section:

$$0 < \gamma \ll \zeta \ll \beta \ll \eta \ll \sigma \ll \phi \ll \psi \ll \varepsilon < 1/6. \tag{12}$$

We will also have that $d$ is a positive integer such that

$$d \le 1/\psi \tag{13}$$

**Lemma 4.3.** *Assuming* (12)*, if* $G = (V, E)$ *is a graph on* $n$ *vertices where* $\delta(G) \geq (1/2+\varepsilon)n$*, then there exists a* $(\phi, \psi, 6)$*-linked partition* $\mathcal{M} = \{V_1, \ldots, V_d\}$ *of* $V$ *for some* $d \leq 1/\psi$*.*

*Proof.* Let $F$ be the bipartite graph with parts $E$ and $V$ such that $ev \in E(F)$ if $ev$ is a triangle in $G$. By the same arguments as in the proof of Proposition 4.2, we have that, for every $v \in V$, $d_F(v) \geq \varepsilon|E|$ and, for every $e \in E$, $d_F(e) \geq 2\varepsilon|V|$.

Therefore, by Lemma 4.1, there exists a pairwise disjoint collection of vertex sets $\{V_1', \ldots, V_d'\}$ such that if $R' := V \setminus \left(\bigcup_{i=1}^d V_i'\right)$, then $|R'| \leq 2\psi n$, and, for every $i \in [d]$, $|V_i'| \geq 2\psi n$ and, for all $i \in [d]$, all but at most $(2\psi)^3 n^2$ pairs $v, v' \in V_i'$ are such that

$$|N_F(v) \cap N_F(v')| \geq (2\psi)^4 n^2. \tag{14}$$

In the remainder of the proof, we will potentially remove some vertices from each of the sets $V_1', \ldots, V_d'$ and then distribute these removed vertices and the vertices in $R'$ into the sets to create the desired partition. To facilitate this, we build an auxiliary graph $H$ with $V(H) = V(G)$ and in which two vertices $v, v' \in V(H)$ are adjacent if and only if $v$ and $v'$ satisfy (14). Also, define $H_i := H[V_i']$ for $i \in [d]$. For any $i \in [d]$, note that

$$N_{H_i}(v) \subseteq L_{4\psi^2,1}(v) \text{ for every } v \in V_i'. \tag{15}$$

Let $J_i := \{v \in V(H_i) : d_{\overline{H_i}}(v) \geq 8\psi^2 n\}$ and $V_i'' := V_i' \setminus J_i$. Since $e(\overline{H_i}) \leq (2\psi)^3 n^2$, we have that

$$|J_i| \leq \frac{8\psi^3 n^2}{8\psi^2 n} = \psi n \text{ and } |V_i''| \geq \psi n \text{ for every } i \in [d].$$

Let $v, v' \in V_i''$. Since $v, v' \notin J_i$,

$$|N_{H_i}(v) \cap N_{H_i}(v')| \geq 2 \cdot (|V_i'| - 8\psi^2 n) - |V_i'| \geq 27\phi n. \tag{16}$$

By (15) and (16), Proposition 3.5 with $k_1 = 1$, $k_2 = 1$, $c = 27\phi$ and $c_1 = c_2 = 4\psi^2$, implies that $v$ and $v'$ are $(9\phi, 2)$-linked. Therefore, $V_i''$ is $(9\phi, 2)$-linked. Similarly, Proposition 3.5 also implies that $V_i''$ is $(3\phi, 3)$-linked and $(\phi, 6)$-linked.

Let $v \in J_1 \cup \cdots \cup J_d \cup R'$. By Proposition 4.2, there exists $i \in [d]$ such that

$$\left|\{u \in V_i'' \colon u \text{ and } v \text{ are } (\varepsilon^2, 1)\text{-linked}\}\right| \geq \frac{\frac{3}{2}\varepsilon^2 n - |R'|}{d} - |J_i| \geq 9\phi n. \tag{17}$$

Therefore, we can construct a partition (that may contain empty parts) $\{W_1, \ldots, W_d\}$ of $J_1 \cup \cdots \cup J_d \cup R'$ such that for every $i \in [d]$ and every $w \in W_i$, $|L_{\varepsilon^2,1}(w) \cap V_i''| \geq 9\phi n$.

Since $V_i''$ is $(9\phi, 2)$-linked, (17) and Proposition 3.5 imply that, for every $w \in W_i$, $V_i'' + w$ is $(3\phi, 3)$-linked and also $(\phi, 6)$-linked. Therefore, for every two distinct vertices $w_1, w_2 \in W_i$, since $|V_i''| \geq 3\phi n$, Proposition 3.5 implies that $w_1$ and $w_2$ are $(\phi, 6)$-linked. Hence, if $V_i := V_i'' \cup W_i$ for every $i \in [d]$, then $\mathcal{M} := \{V_1, \ldots, V_d\}$ is a $(\psi, \phi, 6)$-linked partition of $V$. $\square$

Informally, given a linked partition $\mathcal{M}$ of $V$ and its absorber $A$, we will construct a collection of sets $\mathcal{N}$ that is absorbable with respect to $A$. This set $A$ and the collection $\mathcal{N}$

correspond to $A$ and $\mathcal{N}$ in the discussion at the end of Section 2. As in the discussion, we aim to show that if $U := A \cup V(\mathcal{N})$, then for any small subset $W \in V \setminus U$, with order divisible by 3, $G[U \cup W]$ has a triangle factor. Recall that we will construct this factor by first finding a triangle tiling that covers $W$ in which every triangle has exactly one vertex in $W$ and two vertices in $V(\mathcal{N})$. The second step of this construction is to remove triangles so that there are an equal number of vertices that remain in each of $X(I,1)$, $X(I,2)$ and $X(I,3)$, for every $I \in F_\phi$. We introduce the following definition and straightforward proposition to facilitate the second step of this construction. In Lemma 4.6, we establish all of the important facts about $\mathcal{N}$ that we will need to complete this second step. After the second step, we use the fact that $\mathcal{N}$ is absorbable with respect to $A$ to complete the triangle factor of $G[U \cup W]$.

**Definition 4.4.** Let $G$ be an $n$-vertex graph. Let $\mathcal{S} := \{S_1, \dots, S_m\}$ be a family of subsets of $V(G)$. For $a > 0$, let $C(\mathcal{S}, a)$ be the graph with vertex set $\mathcal{S}$ where the following holds:

$$S_i S_j \in E(C(\mathcal{S}, a)) \iff \begin{array}{l} |\{v \in S_i : |N(v) \cap S_j| \geq an\}| \geq an \text{ and} \\ |\{v \in S_j : |N(v) \cap S_i| \geq an\}| \geq an. \end{array} \tag{18}$$

**Proposition 4.5.** *Let $0 < \gamma < a < 1$ and $d \in \mathbb{N}$. Let $G = (V, E)$ be a graph on $n$ vertices such that $\alpha(G) \leq \gamma n$, $\mathcal{S} = \{S_1, \dots, S_d\}$ a collection of pairwise disjoint subsets of $V$, and $W \subseteq V$ such that $|W| < (a - \gamma)n$. If $P$ is an $(S, S')$-path in $C(\mathcal{S}, a)$, then there exists a set of vertex disjoint triangles $\mathcal{Y}$ in $G[V(\mathcal{S}) \setminus W]$ such that:*

- *$|\mathcal{Y}| = |E(P)|$, $|V(\mathcal{Y}) \cap S| = 1$, $|V(\mathcal{Y}) \cap S'| = 2$ and*
- *$|V(\mathcal{Y}) \cap S''| \in \{0, 3\}$ for every $S'' \in \mathcal{S} - S - S'$.*

*Proof.* Let $S = S_1, \dots, S_\ell = S'$ be $P$. We will iteratively construct vertex disjoint triangles $v_1 e_1, \dots, v_{\ell-1} e_{\ell-1}$, so that $v_i \in S_i \setminus W$ and $e_i \in E(G[S_{i+1} \setminus W])$. We always select $v_i$ so that $d(v_i, S_{i+1}) \geq an$, which is possible by the definition of $C(\mathcal{S}, a)$. Selecting $e_i$ is then possible because $\alpha(G) \leq \gamma n < an - |W|$. $\square$

The following lemma relies heavily on Definitions 3.2, 3.6, 3.7 and 4.4.

**Lemma 4.6.** *For a fixed $k$ and assuming (12), if $G = (V, E)$ is a graph on $n$ vertices such that $\delta(G) \geq (1/2 + \varepsilon)n$, $\mathcal{M} = \{V_1, \dots, V_d\}$ is a $(\psi, \phi, k)$-linked partition of $V$ and $A$ is an $(\mathcal{M}, \phi, \eta)$-absorber such that $|A| \leq \sigma n$, then there exists $\mathcal{N}$ an $(\mathcal{M}, \phi, \eta)$-absorbable collection with respect to $A$ such that:*

*(a) for every $I \in F_\phi(\mathcal{M})$ and $j \in [3]$, $|X(I, j)| = \lfloor \eta n \rfloor$,*

*(b) the graph $C(\mathcal{N}, \beta)$ is connected,*

*(c) for every $v \in V$, there exists $I \in F_\phi(\mathcal{M})$ and $j \in [3]$ such that $d(v, X(I, j)) \geq \beta n$, and*

*(d) for every $I \in F_\phi(\mathcal{M})$, $X(I,1)X(I,2)X(I,3)$ is a triangle in $C(\mathcal{N}, \beta)$.*

*Proof.* Chose $\tau$ so that $\sigma \ll \tau \ll \phi$, and define $V_i' := V_i \setminus A$ for every $i \in [d]$. For every $i, i' \in [d]$, let

$$U_{(i,i')} := \{v \in V_i' : d(v, V_{i'}) \geq \tau n\}.$$

Note that if $V_i$ and $V_i'$ are adjacent in $C(\mathcal{M}, \tau)$, then, by the definition of $C(\mathcal{M}, \tau)$ and the fact that $|A| \leq \sigma n$, we have that $|U_{(i,i')}|, |U_{(i',i)}| \geq \tau n - |A| \geq \tau n/2$.

We first establish the following three simple claims.

**Claim 4.7.** *For every $i \in [d]$, $t_\phi(\mathcal{M}, i) \geq 1$.*

*Proof.* Assume that $t_\phi(\mathcal{M}, i) = 0$. Then the number of triangles containing vertices of $V_i$ is less than $d^2\phi n^3$, but there are at least $\left(\sum_{v \in V_i} e(G[N(v)])\right)/3 \geq \psi n \cdot \varepsilon n^2 \cdot 1/3$ such triangles, a contradiction. □

**Claim 4.8.** *If $I \in F_\phi(\mathcal{M})$ where $\{i, i', i''\} = I$, then $|U_{(i,i')}| \geq \tau n/2$.*

*Proof.* Note that if $\nu_I(i) \geq 2$, then it could be that $i = i'$. Since $I \in F_\phi(\mathcal{M})$, there are at least $\phi n^2$ edges with one end in $V_i$ and the other end in $V_{i'}$, therefore, since $d_G(v, V_{i'}) \leq |V_{i'}| \leq n$, for every $v \in U_{(i,i')}$,

$$\begin{aligned} |U_{(i,i')}| &\geq e_G(U_{(i,i')}, V_{i'})/n \\ &= \left(e_G(V_i, V_{i'}) - e(V_i \setminus U_{(i,i')}, V_{i'})\right)/n \geq \left(\phi n^2 - \tau n \cdot (|V_i| - |U_{(i,i')}|)\right)/n \geq \tau n/2. \end{aligned}$$ □

**Claim 4.9.** *The graph $C(\mathcal{M}, \tau)$ is connected.*

*Proof.* We can assume $d \geq 2$, so let $\mathcal{C}_1, \mathcal{C}_2$ be an arbitrary partition of $\mathcal{M}$ and let $U_i := \bigcup \mathcal{C}_i$ for $i \in [2]$. Without loss of generality we can assume that $|U_1| \leq |U_2|$, so $|U_1| \leq n/2$. We will show that there is an edge in $C(\mathcal{M}, \tau)$ between the sets $\mathcal{C}_1, \mathcal{C}_2$, which will prove the claim. We can assume that $V_1 \in \mathcal{C}_1$. For every $v \in V_1$, we have $|N_G(v) \cap (V \setminus U_1)| \geq \delta(G) - |U_1| \geq \varepsilon n$, so

$$e_G(V_1, U_2) \geq |V_1| \cdot \varepsilon n.$$

Hence, there exists some $V_i \notin \mathcal{C}_2$, say $V_2$, such that $e_G(V_1, V_2) \geq |V_1| \cdot \varepsilon n/d$. For $i \in [2]$, let $x_i$ be the number of vertices in $v \in V_i$ such that $|N_G(v) \cap V_{3-i}| \geq \tau n$. We have the following inequality,

$$x_i \cdot |V_{3-i}| + (|V_i| - x_i) \cdot \tau n \geq |V_1| \cdot \varepsilon n/d.$$

Since $\psi n \leq |V_1|, |V_2| \leq n$ and $\psi\varepsilon/d \geq \psi^2\varepsilon \geq 2\tau$, we have

$$x_i \geq \frac{|V_1| \cdot \varepsilon n/d - |V_i| \cdot \tau n}{|V_{3-i}| - \tau n} \geq \frac{(\psi\varepsilon/d - \tau)n^2}{n} \geq \tau n,$$

which means that $V_1$ and $V_2$ are adjacent in $C(\mathcal{M}, \tau)$. □

Now we proceed to prove Lemma 4.6. For every $i \in [d]$, let the collection $\mathcal{U}_i$ contain the sets $N(v) \cap V'_i$ for every $v \in V(G)$ and $U_{(i,i')}$ for every $i' \in [d]$. Note that $|\mathcal{U}_i| = n + d$ and that every set $U \in \mathcal{U}_i$ is is a subset of $V'_i$.

We will use the following probabilistic argument to contruct the desired $(\mathcal{M}, \phi, \eta)$-absorbable collection $\mathcal{N}$. Let $m := \lfloor \eta n \rfloor$ and select a set $Z_i \subseteq V'_i$ of size $t_\phi(\mathcal{M}, i) \cdot m$ uniformly at random. Then uniformly at random select a partition of $Z_i$ into $t_\phi(\mathcal{M}, i)$ parts each of size $m$ over all such partitions. Note that any such partition corresponds to an $(\mathcal{M}, \phi, \eta)$-absorbable collection, since for every $I \in F_\phi(\mathcal{M})$ and $j \in [3]$, we can uniquely assign $X(I, j)$ to one of the $t_\phi(\mathcal{M}, k(I, j))$ parts of $Z_{k(I,j)}$. Assume there exists such a fixed assignment for every such collection. For any $I \in F_\phi(\mathcal{M})$, $j \in [3]$ and $U \in \mathcal{U}_{k(I,j)}$, the random variable $|U \cap X(I, j)|$ is hypergeometrically distributed[1] and

$$\mathbb{E}|U \cap X(I, j)| = \frac{m}{|V'_{k(I,j)}|} \cdot |U| \geq 0.9 \cdot \eta \cdot |U|.$$

For any $I \in F_\phi(\mathcal{M})$, $j \in [3]$, and any $U \in \mathcal{U}_{k(I,j)}$, when $|U| < \beta n$ the following estimate is trivially true and when $|U| \geq \beta n$ it is implied by the Chernoff bound for the hypergeometric distribution:

$$\mathbb{P}(|U \cap X(I, j)| < \mathbb{E}|U \cap X(I, j)| - \beta n) \leq 2\exp(-\beta^2/3 \cdot \mathbb{E}|U \cap X(I, j)|) \leq 2\exp(-\beta^3 n/3).$$

Hence, by the union bound, w.h.p.

$$|U \cap X(I, j)| \geq \mathbb{E}|U \cap X(I, j)| - \beta n$$

for each of the $n + d$ sets $U \in \mathcal{U}_{k(I,j)}$ simultaneously. Finally, this with the union bound again implies that there exists an $(\mathcal{M}, \phi, \eta)$-absorbable collection $\mathcal{N}$ such that, for each of the at most $d^3 \leq 1/\phi^3$ elements $I$ in $F_\phi(\mathcal{M})$ and every $j \in [3]$,

$$|U \cap X(I, j)| \geq 0.9 \cdot \eta \cdot |U| - \beta n \text{ for every } U \in \mathcal{U}_{k(I,j)}.$$

Rewriting this, we have that, for every $i' \in [d]$, $I \in F_\phi(\mathcal{M})$ and $j \in [3]$,

$$d(v, X(I, j)) \geq 0.9 \cdot \eta \cdot d(v, V'_{k(I,j)}) - \beta n \text{ for every } v \in V, \tag{19}$$

and

$$|U_{(k(I,j),i')} \cap X(I, j)| \geq 0.9 \cdot \eta \cdot |U_{(k(I,j),i')}| - \beta n. \tag{20}$$

For any $I, I' \in F_\phi(\mathcal{M})$ and $j, j' \in [3]$, (19) and (20) imply that

$$\begin{aligned} &\text{if } k(I, j) \neq k(I', j') \text{ and } V_{k(I,j)} V_{k(I',j')} \in E(C(\mathcal{M}, \tau)), \text{ then} \\ &X(I, j) X(I', j') \in E(C(\mathcal{N}, \beta)). \end{aligned} \tag{21}$$

[1]That is, if we have a bin with $|V_{k(I,j)}|$ balls and exactly $|U|$ of them are red, then the probability that there are exactly $t$ red balls after drawing $m$ balls without replacement from the bin is $\mathbb{P}(|U \cap X(I, j)| = t)$.

Also note that Claim 4.7 implies that,

$$\text{for every } i \in [d], \text{ there exists } I \in F_\phi(\mathcal{M}) \text{ and } j \in [3] \text{ such that } X(I,j) \subseteq V_i. \tag{22}$$

Combining (21) and (22), we have that for any $I, I' \in F_\phi(\mathcal{M})$ and $j, j' \in [3]$, if $k(I,j) \neq k(I',j')$ and there is a path from $V_{k(I,j)}$ to $V_{k(I',j')}$ in $C(\mathcal{M},\tau)$, then there is a path from $X(I,j)$ to $X(I',j')$ in $C(\mathcal{N},\beta)$. This, (22) and Claim 4.9 imply that when $d \geq 2$, the graph $C(\mathcal{N},\beta)$ is connected. Also, for all $d \geq 1$, (21) and Claim 4.8, imply that $X(I,1)X(I,2)X(I,3)$ is a triangle in $C(\mathcal{N},\beta)$ for every $I \in F_\phi(\mathcal{M})$. Therefore, (d) holds. This and Claim 4.7, imply that $C(\mathcal{N},\beta)$ is isomorphic to $K_3$ when $d = 1$, so (b) holds for all $d \geq 1$. Since (a) is true by construction, only (c) remains to be proved. To see that (c) holds, note that for every $v \in V$, there exists $i \in [d]$ such that $d(v, V_i') \geq ((1/2+\varepsilon)n - |A|)/d \geq \phi n$. Since (22) implies that there exist $I \in F_\phi(\mathcal{M})$ and $j \in [3]$ such that $X(I,j) \subseteq V_i$, (19) implies that $d(v, X(I,j)) \geq \beta n$. □

*Proof of Lemma 2.1.* Assume (12) holds. Lemma 4.3 implies that there exists a $(\psi,\phi,6)$-linked partition $\mathcal{M}$ of $V$. Lemma 3.8 implies that there exists $A \subseteq V$ such that $|A| \leq \sigma n$ and $A$ is an $(\mathcal{M},\phi,\eta)$-absorber. Lemma 4.6 then implies that there exists a collection $\mathcal{N}$ of disjoint subsets of $V \setminus A$ such that $\mathcal{N}$ is $(\mathcal{M},\phi,\eta)$-absorbable with respect to $A$ and that properties (a), (b), (c) and (d) of Lemma 4.6 hold. Let $N := V(\mathcal{N})$, i.e. $N := \bigcup\{X(I,j) : I \in F_\phi, j \in [3]\}$. Let $U := A \cup N$ and $W \subseteq V \setminus U$ be such that $|W| \leq \zeta n$ and $|W|$ is divisible by 3. We have that $|U| \leq \sigma n + 3d^3\eta n \leq 2\sigma n$ and we will show that there is a triangle factor of $G[W \cup U]$ which will complete the proof.

For every $w \in W$, by Lemma 4.6(c), there exists some $I \in F_\phi$ and $j \in [3]$, such that $d(w, X(I,j)) \geq \beta n > \gamma n + 2|W|$. Therefore, since $\alpha(G) \leq \gamma n$, for every $w \in W$, we can assign some edge $e_w \in E(G[N(w) \cap X(I,j)])$ to $w$ so that $\mathcal{W} := \{we_w : w \in W\}$ is a collection of vertex disjoint triangles.

The idea of the remainder of the proof is the following. We iteratively construct another small collection $\mathcal{Y}$ of vertex disjoint triangles in $G[N \setminus V(\mathcal{W})]$. For convenience, we will use $\mathcal{Y}$ to represent the triangles that have been constructed so far in this iterative process. In particular, at the beginning of this process $\mathcal{Y} = \emptyset$. For every $I \in F_\phi$ and $j \in [3]$, we define $X'(I,j) := X(I,j) \setminus V(\mathcal{W} \cup \mathcal{Y})$. We also define $\mathcal{N}' := \{X'(I,j) : I \in F_\phi, j \in [3]\}$ and $N' := V(\mathcal{N}')$. After this process is completed and we have finished constructing $\mathcal{Y}$, we will have that, for every $I \in F_\phi$, $|X'(I,1)| = |X'(I,2)| = |X'(I,3)|$. Note that then because $A$ is an $(\mathcal{M},\phi,\eta)$-absorber, and Lemma 4.6(a) implies that $|X'(I,1)| = |X'(I,2)| = |X'(I,3)| \leq \eta n$, the collection $\mathcal{N}'$ is $(\mathcal{M},\phi,\eta)$-absorbable with respect to $A$, so there exists a triangle factor $\mathcal{Z}$ of $G[A \cup N']$. Therefore, $\mathcal{W} \cup \mathcal{Y} \cup \mathcal{Z}$ is a triangle factor of $G[W \cup A \cup N] = G[W \cup U]$, which completes the proof.

We will now describe the two stage process for constructing $\mathcal{Y}$. Our goal in the first stage is for the following to hold for every $I \in F_\phi$:

$$|X'(I,1) \cup X'(I,2) \cup X'(I,3)| \equiv 0 \pmod 3. \tag{23}$$

At any step of the first stage of the algorithm, we call a triple $I \in F_\phi$, *bad* if it does not satisfy (23). Pick a bad $I \in F_\phi$ such that $|X'(I,1) \cup X'(I,2) \cup X'(I,3)| \equiv 1 \pmod 3$ if possible. Note that $|N'|$ is always divisible by 3, because $|N'| = |N| - 2|W| - 3|\mathcal{Y}|$ and both $|W|$ and $|N|$ are divisible by 3. Therefore, there exists another bad triple $I' \in F_\phi - I$. By Lemma 4.6(b) there exists a path $P$ from $X(I,1)$ to $X(I',1)$ in the graph $C(\mathcal{N}, \beta)$. Hence, by Proposition 4.5, we can add a collection of at most $|P|-1$ vertex disjoint triangles to $\mathcal{Y}$, so that after this step, at least one of $I$ or $I'$ is no longer bad and every triple in $F_\phi$ that was good before this step remains good after this step is completed. Note that we finish the first phase in at most $|\mathcal{N}|$ steps, so $|\mathcal{Y}| \le |\mathcal{N}|(|\mathcal{N}|-1) \le (3 \cdot d^3)^2$ after the first phase.

In each step of the second and final stage of the algorithm, we pick some $I \in F_\phi$ such that $|X'(I,1)| = |X'(I,2)| = |X'(I,3)|$ does not hold and add triangles contained in $G[X(I,1) \cup X(I,2) \cup X(I,3)]$ to $\mathcal{Y}$ until $|X'(I,1)| = |X'(I,2)| = |X'(I,3)|$ holds. We continue in this manner until we have the desired collection $\mathcal{Y}$. We will now describe this process for a fixed $I \in F_\phi$. Before each triangle is constructed, we relabel $\{j_1, j_2, j_3\} = [3]$ so that $|X'(I,j_1)| \le |X'(I,j_2)| \le |X'(I,j_3)|$ and let

$$c(I) := (|X'(I,j_2)| - |X'(I,j_1)|) + (|X'(I,j_3)| - |X'(I,j_1)|).$$

We also fix $\Phi := c(I)$ before any triangle is constructed. Because $|\mathcal{Y}| \le 9 \cdot d^6$ at the start of the second stage of the algorithm, $|\mathcal{W}| = |W|$, and every triangle in $\mathcal{Y} \cup \mathcal{W}$ has at most 2 vertices in $X(I,j)$ for any $j \in [3]$, we have that

$$\Phi \le 2 \cdot 2(9 \cdot d^6 + |W|) < 2\zeta n.$$

Note that because $I$ satisfies (23), we can conclude that $\Phi \equiv c(I) \equiv 0 \pmod 3$ throughout this process.

We now add a triangle to $\mathcal{Y}$ with one vertex in $X(I,j_2)$ and two vertices in $X(I,j_3)$ until $c(I) = 0$, which implies $|X'(I,1)| = |X'(I,2)| = |X'(I,3)|$. Recall that if $|X'(I,j_1)| = |X'(I,j_2)|$ or $|X'(I,j_2)| = |X'(I,j_3)|$, then, after adding the triangle to $\mathcal{Y}$, we relabel $\{j_1, j_2, j_3\} = [3]$, so that we again have that $|X(I,j_1)| \le |X(I,j_2)| \le |X(I,j_3)|$. By Lemma 4.6(d), $X(I,1)X(I,2)X(I,3)$ is a triangle in $C(\mathcal{N}, \beta)$. Therefore, there exists $v \in X'(I,j_2)$ such that $d(v, X'(I,j_3)) > \gamma n = \alpha(G)$ and, hence, a triangle with one vertex in $X'(I,j_2)$ and two vertices in $X'(I,j_3)$, provided

$$|V(\mathcal{Y} \cup \mathcal{W}) \cap X(I,j_2)|, |V(\mathcal{Y} \cup \mathcal{W}) \cap X(I,j_3)| < (\beta - \gamma)n. \tag{24}$$

Assuming (24) always holds, this process will terminate after constructing at most $2 \cdot \Phi/3$ triangles, because $c(I)$ decreases by 3 after each triangle is added to $\mathcal{Y}$ unless $|X'(I,j_1)| = |X'(I,j_2)|$, and when $|X'(I,j_1)| = |X'(I,j_2)|$, $c(I)$ does not change, but $c(I)$ decreases by 3 when the following triangle is added to $\mathcal{Y}$. Therefore, $V(\mathcal{Y} \cup \mathcal{W})$ intersects any set in $\mathcal{N}$ in at most $2(2 \cdot \Phi/3 + 9 \cdot d^6 + |W|) < (\beta - \gamma)n$ vertices. Hence, (24) always holds and we can find the required triangles between $X'(I,j_2)$ and $X'(I,j_3)$.

□

## 4.2 Proof of Lemma 2.2

*Proof of Lemma 2.2.* Set $\gamma < \varepsilon/36$. Let $\mathcal{T}$ be a maximum family of disjoint triangles in $G$, and $\mathcal{M}$ be a maximum matching in $G[V \setminus V(\mathcal{T})]$. Denote $\mathcal{V}$ the set of remaining vertices and let $v = |\mathcal{V}|$, i.e. $v = |V \setminus V(\mathcal{T} \cup \mathcal{M})|$. Denote $t := |\mathcal{T}|$ and $m := |\mathcal{M}|$, then we have $n = 3t + 2m + v$, $v \leq \alpha(G) \leq \gamma n$ and $t \geq (\delta(G) - \alpha(G))/3 \geq n/6$ by greedy construction.

**Claim 4.3.** *$m < 8/\varepsilon$.*

*Proof.* For a contradiction, assume $\varepsilon m \geq 8$. Note that for every vertex $u \in V(\mathcal{M})$, its degree in $G[V \setminus V(\mathcal{T})]$ is at most $v + m$, otherwise $u$ is adjacent to both ends of a matching edge in $\mathcal{M}$, contradicting the maximality of $\mathcal{T}$. Thus

$$\begin{aligned} d(u, V(\mathcal{T})) &\geq \left(\frac{1}{2} + \varepsilon\right) n - v - m = \left(\frac{1}{2} + \varepsilon\right)(3t + 2m + v) - v - m \\ &\geq \left(\frac{3}{2} + 3\varepsilon\right) t + 2\varepsilon m - \frac{v}{2} \geq \left(\frac{3}{2} + \varepsilon\right) t, \end{aligned}$$

where the last inequality follows from the fact that $v \leq \gamma n$ and $t \geq n/6$. Thus

$$e(V(\mathcal{M}), V(\mathcal{T})) \geq \left(\frac{3}{2} + \varepsilon\right) t \cdot 2m = (3 + 2\varepsilon)tm. \tag{25}$$

Let $\mathcal{T}'$ be the collection of triangles in $\mathcal{T}$, each sending at least $3m + 9$ edges to $\mathcal{M}$ and write $t' = |\mathcal{T}'|$. Note that each triangle $T \in \mathcal{T}$ can send at most $6m$ edges to $\mathcal{M}$, thus

$$e(V(\mathcal{M}), V(\mathcal{T})) \leq t' \cdot 6m + (t - t')(3m + 8) = (3m + 8)t + (3m - 8)t'.$$

Together with (25) we have that

$$t' \geq \frac{2\varepsilon m - 8}{3m - 8} \cdot t \geq \frac{\varepsilon m}{3m - 8} \cdot t \geq \frac{\varepsilon}{3} \cdot t \geq \frac{\varepsilon n}{18}. \tag{26}$$

Note that for every $T \in \mathcal{T}'$, there is at least one vertex $s_T \in V(T)$ that sends at least $(3m + 9)/3 = m + 3$ edges to $\mathcal{M}$. Hence, $s_T$ forms a triangle with at least 3 edges in $\mathcal{M}$. Let $S := \{s_T : T \in \mathcal{T}'\}$ and $R := V(\mathcal{T}') \setminus S$.

By the definition of $\mathcal{T}'$, we have $e(V(\mathcal{M}), V(\mathcal{T}')) \geq (3m + 9)t'$. Thus there exists $u \in V(\mathcal{M})$ such that

$$d(u, V(\mathcal{T}')) \geq \frac{e(V(\mathcal{M}), V(\mathcal{T}'))}{2m} \geq \frac{(3m + 9)t'}{2m} \geq \frac{3t'}{2}.$$

With (26) we have that $d(u, V(\mathcal{R})) \geq d(u, V(\mathcal{T}')) - |S| \geq t'/2 \geq (\varepsilon n)/36 > \gamma n$.

Since $\alpha(G) \leq \gamma n$, there is at least one edge $y_1 y_2 \in N_R(u)$. Let $T$ be the triangle $u y_1 y_2$ and let $T_1, T_2 \in \mathcal{T}$ such that $y_i \in T_i$ for $i \in [2]$. Since, for $i \in [2]$, $s_{T_i}$ forms a triangle with at least three edges in $\mathcal{M}$, we can pick distinct edges in $e_1, e_2 \in \mathcal{M}$ such that neither contains $u$ and $s_{T_i} e_i$ is a triangle for $i \in [2]$. If $T_1 \neq T_2$, then $\mathcal{T} - T_1 - T_2 + T + s_{T_1} e_1 + s_{T_2} e_2$ contradicts the maximality of $\mathcal{T}$, and if $T_1 = T_2$, then $\mathcal{T} - T_1 + T + s_{T_1} e_1$ contradicts the maximality of $\mathcal{T}$. □

**Claim 4.4.** *$v \leq 1$.*

*Proof.* Suppose on the contrary that there exist two distinct vertices $x, y \in V(\mathcal{V})$. Because $V(\mathcal{V})$ is an independent set, we have that $v \leq \gamma n$, and, by Claim 4.3, $m < 8/\varepsilon$, therefore

$$e(\{x,y\}, V(\mathcal{T})) \geq 2(\delta(G) - m) \geq (1+\varepsilon)n > 3t + \varepsilon n.$$

Denote $\mathcal{T}'' := \{T \in \mathcal{T} : e(\{x,y\}, T) \geq 4\}$. It follows that $t'' := |\mathcal{T}''| \geq \varepsilon n/3 > \gamma n$. Fix a triangle $T = abc \in \mathcal{T}''$. If $d(x, V(T)) = 3$ and $d(y, V(T)) = 1$, say $ya \in E(G)$, then the existence of the triangle $xbc$ and the edge $ya$ contradict the maximality of $\mathcal{M}$. Thus we may assume that $d(x, V(T)) = d(y, V(T)) = 2$. Note that if $x$ is adjacent to $\{a, b\}$ and $y$ is adjacent to $\{a, c\}$, then the existence of the triangle $xab$ and the edge $yc$ contradict the maximality of $\mathcal{M}$. Hence, both $x$ and $y$ are adjacent to the same two vertices in $T$. Let $S := N(x) \cap N(y) \cap V(\mathcal{T}'')$, and $R := V(\mathcal{T}'') \setminus S$. Since $|R| = t'' > \gamma n$, there exist two triangles $abc, a'b'c' \in \mathcal{T}''$ such that $cc' \in E(G[R])$. Now the existence of the triangles $xab$ and $ya'b'$ and the edge $cc'$, contradict the maximality of $\mathcal{M}$. □

The number of vertices not covered in $\mathcal{T}$ is then $2m + v < 16/\varepsilon + 1$.

□

# 5 Proof of Theorem 1.3

We now complete the proof of Theorem 1.3, by proving Lemma 2.3 and Lemma 2.4. First we introduce some special notation that is used only in this section.

**Notation.** For disjoint vertices $x$,$y$, and $z$, we will let $x$, $xy$ and $xyz$ represent the sets $\{x\}$, $\{x, y\}$ and $\{x, y, z\}$, respectively. It should be clear from context whether we mean for $x$ to represent the vertex $x$ or the singleton set $\{x\}$. For any $U \subseteq V$, we will let $\overline{U} = V \setminus U$, $\|U\| := \sum_{e \in \binom{U}{2}} w(e) \cdot 3$ and for $W \subseteq \overline{U}$ we will let $\|U, W\| := \sum_{e \in E(U,W)} w(e) \cdot 3$. For disjoint vertices $x, y$ and $z$ we call $xyz$ a *$t$-heavy triangle* if $\|xyz\| > 9t$. We multiply by three here purely for notational convenience.

To prove the absorbing lemma, we will consider the very simple partition $\mathcal{M} := \{V_1\}$ of $V$, i.e. $V_1 := V$. We show that there are at least $\phi n^3$ $t$-heavy triangles in $G$, i.e. $t_\phi(\mathcal{M}, \{1, 1, 1\}) = 1$ and that the entire vertex set $V_1 := V$ is $(\phi, 1)$-linked. Applying Lemma 3.8 will essentially complete the proof of the absorbing lemma.

*Proof of Lemma 2.3.* Let $\sigma \ll \phi \ll \varepsilon$. Claim 5.1 and Claim 5.2 make up the bulk of the proof.

**Claim 5.1.** *There are at least $\frac{1}{4}n^3$ ordered triples $(x, y, z) \in V^3$ such that $xyz$ is a $t$-heavy triangle.*

*Proof.* Pick any $x \in V$. For any $y \in V - x$, let $V' = V - x - y$ and

$$Z_y := \{z \in V' : xyz \text{ is } t\text{-heavy triangle}\}.$$

If $\|xy\| \leq 9t-6$, then $|Z_y| = 0$. So, since $t < 1$,

$$|Z_y| = 0 \geq \frac{\|xy\| - 9t + 6}{9(1-t)}|V'| > \frac{\|xy\| - 5t + 2}{9(1-t)}|V'|.$$

Otherwise, because $\delta_w(G) \geq \left(\frac{1+2t}{3} + \varepsilon\right) n$,

$$(2+4t)|V'| < \|xy, V'\| \leq 6 \cdot |Z_y| + (9t - \|xy\|)|V' \setminus Z_y| = (6 - 9t + \|xy\|)\,|Z_y| + (9t - \|xy\|)|V'|,$$

so, since $|Z_y| \geq 0$ and $\|xy\| \leq 3$,

$$|Z_y| \geq \frac{\|xy\| - 5t + 2}{6 - 9t + \|xy\|}|V'| \geq \frac{\|xy\| - 5t + 2}{9(1-t)}|V'|.$$

Therefore, using both bounds for $|Z_y|$, there are at least

$$\sum_{y \in V - x} |Z_y| \geq \sum_{y \in V - x} \frac{\|xy\| - (5t-2)}{9(1-t)} \cdot |V'| > \frac{1 + 2t - (5t-2)}{9(1-t)} \cdot (n-2)^2 = \frac{1}{3}(n-2)^2$$

pairs $(y, z)$ such that $xyz$ is a $t$-heavy triangle, and this completes the proof. □

**Claim 5.2.** *For every pair of distinct vertices $x$ and $y$ there are at least $2\phi^2 n^2$ ordered pairs $(z, w) \in (V - x - y)^2$ such that $xzw$ and $yzw$ are both $t$-heavy triangles.*

*Proof.* Assume the contrary. For $0 \leq c \leq 6$, let

$$Z_c := \{z \in V - x - y : \|z, xy\| > c\}.$$

For any $z \in V$ we will say $w \in V - xyz$ *works* with $z$ if both $xzw$ and $yzw$ are $t$-heavy triangles.

First note that if $z \in V - x - y$ is such that $\|xz\|, \|yz\| > 3t$, then any vertex $w \in V - x - y - z$ such that $\|w, xyz\| \geq 3 + 6t$ works with $z$.

If $z \in Z_{3+3t}$, then, because $\|xyz, V \setminus xyz\| > (3 + 6t + 9\varepsilon)n - 2\|xyz\|$, there are at least $2\phi n$ vertices $w$ such that $\|w, xyz\| > 3 + 6t$. By the previous observation, every such $w$ works with $z$. Therefore, we can assume that $|Z_{3+3t}| < \phi n$.

Since $\|xy, V \setminus xy\| \geq (2 + 4t + 6\varepsilon)n - 2\|xy\|$, we have $|Z_{2+4t}| \geq 2\phi n$. Therefore, if for every vertex in $z \in Z_{2+4t}$ there are $\phi n$ vertices that work with $z$, then we are done. Assume that this is not the case, and let $z \in Z_{2+4t}$ be such that there are fewer than $\phi n$ vertices that work with $z$. From previous arguments, we have that $z \notin Z_{3+3t}$. Let $G^*$ be the graph obtained from $G$ by removing the vertices in $Z_{3+3t}$ and the vertices that work with $z$ from $G$. Note that we removed at most $2\phi n$ vertices, so $G^*$ has the following properties:

$$\text{(a) } Z_{3+3t} = \emptyset, \text{ (b) no vertices work with } z, \text{ and (c) } \delta_w(G^*) \geq \left(\frac{1+2t}{3} + \frac{\varepsilon}{2}\right) n. \tag{27}$$

Assume without loss of generality that $\|xz\| \geq \|yz\|$.

Let $V' := V(G^*) \setminus xyz$, $Y := \{w \in V' : yzw \text{ is a } t\text{-heavy triangle}\}$ and $X := V' \setminus Y$.

By (27)(c), there exists $w \in V'$ such that $\|w, xyz\| \geq 3 + 6t$. If $\|xz\| \geq \|yz\| > 3t$, then, because $\|xw\|, \|yw\| \leq 3$, we have that

$$\|w, yz\|, \|w, xz\| \geq \|w, xyz\| - 3 \geq 6t.$$

Therefore, both $\|xzw\|$ and $\|yzw\|$ are $t$-heavy triangles, contradicting (27)(b). Hence $\|yz\| \leq 3t$, which implies that if we let $\bar{t} := 1 - t$ and $c := (\|yz\| - 3t) + \bar{t} = \|yz\| - (3 - 4\bar{t})$, then $c \leq \bar{t}$. Because $z \in Z_{2+4t} = Z_{6-4\bar{t}}$, it must be that

$$\|xz\| > 3 - c, \tag{28}$$

so $c > 0$. Combining the upper and lower bounds on $c$ gives

$$\bar{t} \geq c > 0. \tag{29}$$

Note that

$$w \in Y \text{ if and only if } \|w, yz\| > 6t + \bar{t} - c = 5t + 1 - c. \tag{30}$$

Therefore, we have that

$$\|yz, V'\| < 3|Y| + \|z, Y\| + (5t+1-c)|X| = \|z, Y\| + (5\bar{t} - 3 + c)|Y| + (5t+1-c)|V'|, \tag{31}$$

and, by (27)(c),

$$\|yz, V'\| \geq (2 + 4t)|V'| = \left(\bar{t} + c\right)|V'| + (5t + 1 - c)|V'|. \tag{32}$$

If $w \in X$, then (30) implies $\|w, xyz\| < 5t + 4 - c$. If $w \in Y$ and $\|w, xyz\| \geq 9t + c$, then (28) implies that

$$\|xzw\| \geq \|w, xyz\| - \|wy\| + \|xz\| > (9t + c) - 3 + (3 - c) = 9t,$$

which contradicts (27)(b). Combining this with (27)(c), implies

$$(3 + 6t)|V'| < \|xyz, V'\| < (5t + 4 - c)|X| + (9t + c)|Y| = (5t + 4 - c)|V'| - (4\bar{t} - 2c)|Y|.$$

Then combining this with the obvious bound $\|z, Y\| \leq 3|Y|$, (31) and (32), imply

$$\frac{\bar{t} - c}{4\bar{t} - 2c} > \frac{|Y|}{|V'|} > \frac{\bar{t} + c}{5\bar{t} + c} \text{ which implies } c^2 - 6\bar{t}c + \bar{t}^2 > 0.$$

With (29), we have that

$$0 \leq c < \bar{t}(3 - 2\sqrt{2}) < \bar{t}/2. \tag{33}$$

Again using the fact that $\|w, xyz\| < 9t + c$ for every vertex $w \in Y$, but this time also using (27)(a), we have that

$$(3 + 3t)|X| + \|z, X\| + (9t + c)|Y| \geq \|xyz, V'\| > (2 + 4t)|V'| + \|z, X\| + \|z, Y\|$$

so

$$0 > \|z, Y\| - \bar{t}|X| + (5\bar{t} - 3 - c)|Y| = \|z, Y\| - \bar{t}|V'| + (6\bar{t} - 3 - c)|Y|.$$

By (31) and (32), $\|z, Y\| - (\bar{t} + c)|V'| + (5\bar{t} - 3 + c)|Y| > 0$, so $c|V'| + (\bar{t} - 2c)|Y| < 0$. This contradicts (33). □

Now we can quickly prove Lemma 2.3. Recall Definitions 3.3, 3.6 and 3.7. Claim 5.1 implies that $V$ is $(1,\phi,1)$-linked, so if we let $V_1 := V$ and $\mathcal{M} = \{V_1\}$, then $\mathcal{M}$ is a $(1,\phi,1)$-linked partition of $V$. Claim 5.2 implies that $t_\phi(\mathcal{M},\{1,1,1\}) = 1$ and $F_\phi(\mathcal{M}) = \{\{1,1,1\}\}$. Now we can apply Lemma 3.8 to $\mathcal{M}$. Let $U$ and $\zeta$ be $A$ and $\eta$ from Lemma 3.8, respectively. The set $U$ is the desired set, since when $W \subseteq V \setminus U$ is such that $|W|$ is at most $\zeta n$ and divisible by 3, any partition of $W$ into three parts each of size $|W|/3$ is $(\mathcal{M},\phi,\eta)$-absorbable with respect to $A$. □

*Proof of Lemma 2.4.* Let $\mathcal{R}$ be a collection of vertex disjoint $t$-heavy triangles in $G$, let $U := V(\mathcal{R})$, $W := V \setminus U$, and $\rho := \sum_{T\in\mathcal{R}} \|T\|$. Let $M \subseteq E(G[U])$ be a matching such that for every $e \in M$, $\|e\| > 3t$, and let $I$ be the set of vertices in $W$ not incident to an edge in $M$. Assume that $\mathcal{R}$ and $M$ are picked to maximize the triple $(|\mathcal{R}|, |M|, \rho)$ lexicographically.

Clearly $|W| = 2|M| + |I|$, so the following two claims complete the proof.

**Claim 5.3.** *$|M| \le 2$.*

*Proof.* Suppose there exist three distinct edges $e_1, e_2, e_3 \in M$. By the maximality of $|\mathcal{R}|$, for $i \in \{1,2,3\}$ and any $x \in W - e_i$, $\|e_i, x\| < 6t$. Therefore, $\|e_1 \cup e_2 \cup e_3, W\| \le 6t|W|$, so

$$\|e_1 \cup e_2 \cup e_3, U\| > 6 \cdot 3\delta_w(G) - 6t|W| > 6 \cdot (1+2t)|U| = (18+36t)|\mathcal{R}|,$$

so there exist $T \in \mathcal{R}$ such that $\|e_1 \cup e_2 \cup e_3, T\| > 18+36t$. Without loss of generality assume that $\|e_1, T\| \ge \|e_2, T\| \ge \|e_3, T\|$.

Since $18 \ge \|e_1, T\| > 6 + 12t$, $\|e_2, T\| > 18t$. Now, label $\{t_1, t_2, t_3\} := V(T)$ so that $\|e_1, t_1\| \ge \|e_1, t_2\| \ge \|e_1, t_3\|$. Since $6 \ge \|e_1, t_1\| > 2 + 4t$, we have that $\|e_1, t_2\| > 6t$, and both $e_1t_1$ and $e_1t_2$ are $t$-heavy triangles. Because $\|e_2, T\| > 18t$, there exists $i \in \{1,2,3\}$ such that $\|e_2, t_i\| > 6t$ which implies $e_2t_i$ is a $t$-heavy triangle. Let $j \in \{1,2\} - i$. Since $e_1t_j$ and $e_2t_i$ are disjoint $t$-heavy triangle, we have violated the maximality of $|\mathcal{R}|$. □

**Claim 5.4.** *$|I| \le 2$.*

*Proof.* Suppose there are disjoint vertices $x_1, x_2, x_3 \in I$ and let $X = \{x_1, x_2, x_3\}$. By the maximality of $|\mathcal{R}|$, $\|x_i, e\| < 6t$ for every $e \in M$ and $i \in [3]$. Furthermore, by the maximality of $|M|$, $\|x_i, y\| \le 3t$ for every $y \in I - x_i$. Therefore, $\|X, W\| \le 9t|W|$ and

$$\|X, U\| > 3 \cdot 3\delta_w(G) - 9t|W| > 3 \cdot (1+2t)|U| = (9+18t)|\mathcal{R}|,$$

so there exists $T \in \mathcal{R}$ such that $\|X, T\| > 9 + 18t$. Without loss of generality assume that $\|x_1, T\| \ge \|x_2, T\| \ge \|x_3, T\|$.

Note that $9 \ge \|x_1, T\| > 3 + 6t$ which implies $\|x_2, T\| > 9t$ and $\|x_2, t_1\| > 3t$ for some $t_1 \in T$. Therefore, by the maximality of $|M|$, to complete the proof we only need to show that $x_1t_2t_3$ is a $t$-heavy triangle where $\{t_2, t_3\} = V(T) - t_1$. For the rest of the proof we will focus on $x_1$ so, for notation simplicity, let us define $x := x_1$.

Now suppose $xt_2t_3$ is not a $t$-heavy triangle, i.e.

$$\|xt_2\| + \|xt_3\| + \|t_2t_3\| \le 9t. \tag{34}$$

Note that for any labeling $\{i,j,k\} = \{1,2,3\}$ since $\|xt_k\| \le 3$, we have $\|x, t_it_j\| > 6t$, so $xt_it_j$ is a $t$-heavy triangle when $\|t_it_j\| \ge 3t$. Therefore, $\|t_2t_3\| < 3t$, and, furthermore, because $t_1t_2t_3$ is a $t$-heavy triangle, we have that $\|t_1t_2\| + \|t_1t_3\| > 6t$. Assume without loss of generality, that $\|t_1t_2\| \ge \|t_1t_3\|$, so $\|t_1t_2\| > 3t$. This implies that $xt_1t_2$ is a $t$-heavy triangle, and, by the maximality of $\rho$,

$$\|xt_1\| + \|xt_2\| \le \|t_1t_3\| + \|t_2t_3\|. \tag{35}$$

Furthermore, since $\|xt_1\| + \|xt_2\| > 6t$ and $\|t_2t_3\| < 3t$, this implies $\|t_1t_3\| > 3t$. Therefore, $xt_1t_3$ is a $t$-heavy triangle, and, again by the maximality of $\rho$,

$$\|xt_1\| + \|xt_3\| \le \|t_1t_2\| + \|t_2t_3\|. \tag{36}$$

By (34), $\|t_2t_3\| \le 9t - (\|xt_2\| + \|xt_3\|)$. Combining this with (35) and (36), we get that

$$2\|xt_1\| + \|xt_2\| + \|xt_3\| \le \|t_1t_2\| + \|t_1t_3\| + 18t - 2(\|xt_2\| + \|xt_3\|).$$

Hence,

$$\|xt_2\| + \|xt_3\| + 2\|x, T\| \le \|t_1t_2\| + \|t_1t_3\| + 18t.$$

This is a contradiction, because

$$\|xt_2\| + \|xt_3\| + 2\|x, T\| > 6t + 2(3 + 6t) = 6 + 18t \text{ and } \|t_1t_3\| + \|t_1t_2\| + 18t \le 6 + 18t.$$

□

# 6 Concluding Remarks

In this paper we answered Question 1.1 for $k = 3$, and it remains open for $k \ge 4$. We now give constructions which show that the minimum degree necessary for the existence of a $K_k$-factor in Question 1.1 is at least $\left(\frac{k-2}{k} + o(1)\right)n$ for every $k \ge 4$. In the following constructions, we call an $n$ vertex triangle-free graph with independence number $o(n)$ and minimum degree $o(n)$ an *Erdős graph*, which we denote by $\mathrm{ER}(n)$.

**Example 6.1.** For the case $k = 2\ell + 1$, consider the complete $(\ell + 1)$-partite graph with one part $V_0$ of size $n/k - 1$, another part $V_1$ of size $2n/k + 1$ and the remaining parts $V_2, \dots, V_\ell$ each of size $2n/k$. To complete the construction, for $i = 0, \dots, \ell$, put a copy of $\mathrm{ER}(|V_i|)$ on the set $V_i$. This graph does not have a $K_k$-tiling, because each $K_k$ has at most 2 vertices in $V_1$ and a $K_k$-tiling can have at most $n/k$ copies of $K_k$. The minimum degree of this graph is $\left(\frac{k-2}{k} + o(1)\right)n$ and it has sublinear independence number. Note that this construction has the additional property of being $K_{k+2}$-free. For the case $k = 2\ell$, start with the complete $\ell$-partite graph with parts $V_1, \dots, V_\ell$ where $V_1$ has size $2n/k + 1$, $V_2$ has size $2n/k - 1$ and the remaining parts each have size $2n/k$, and place $\mathrm{ER}(|V_i|)$ on each of the parts $V_1, \dots, V_\ell$. This again gives a graph with no $K_k$-factor, sublinear independence number and minimum degree $\left(\frac{k-2}{k} + o(1)\right)n$. Note that, in this case, the graph is $K_{k+1}$-free.

Note that large cliques are present in all of our examples which show that the minimum degree condition in Theorem 1.2 is asymptotically sharp. This motivates the following question.

**Question 6.2.** *Let $G$ be an $n$-vertex $K_r$-free graph with $\alpha(G) = o(n)$ for some constant $r \geq 4$. What is the minimum degree condition on $G$ that guarantees a triangle factor in $G$?*

For the case $r = 4$, we use a modified version of the Bollobás-Erdős graph [7] to construct a graph without a triangle factor with high minimum degree. For every large even $n$, the Bollobás-Erdős graph is an $n$-vertex $K_4$-free graph with independence number $o(n)$, which we denote by $\mathrm{BE}(n)$.[2] The vertex set of $\mathrm{BE}(n)$ is the disjoint union of two sets $V_1$ and $V_2$ of the same order such that the graphs $G[V_1]$ and $G[V_2]$ are triangle-free and $d(v_i, V_{3-i}) \geq (1/4 - o(1))n$ for every $v_i \in V_i$. To construct our graph, we start with $\mathrm{BE}(4n/3 + 2)$ and then remove a random subset of size $n/3 + 2$ from one of the two parts. Note that the two parts now have sizes $n/3 - 1$ and $2n/3 + 1$, and, with high-probability, this gives a $K_4$-free graph with minimum degree $(1/6 - o(1))n$ that does not have a triangle factor.

For the case $r = 5$, we can use Example 6.1 with $k = 3$, i.e. just the parts $V_0$ and $V_1$, to show that we need $\delta(G) \geq (1/3 + o(1))n$. Noga Alon commented that if one is only looking for $n/3 - 1$ vertex disjoint triangles, instead of a triangle factor, then maybe the minimum degree condition $(1/3 + o(1))n$ is sufficient (with no condition on the clique number). We believe that we will answer Question 6.2 for $r \geq 5$, and prove the correctness of Alon's comment in a forthcoming manuscript [6].

One can also consider a more general question.

**Question 6.3.** *Let $r, k$ be such that $r > k$, let $G$ be an $n$-vertex $K_r$-free graph with $\alpha(G) = o(n)$. What is the minimum degree condition on $G$ that guarantees a $K_k$-factor in $G$?*

When $k$ is even and $r = k + 1$, Example 6.1 shows that the minimum degree must be at least $(\frac{k-2}{k} + o(1))n$. When $k = 2\ell+1 \geq 5$ and $r = k+1$, we can modify the construction above slightly by having one part $V_0$ of size $3n/k+1$ one part $V_1$ of size $2n/k-1$, and parts $V_2, \ldots, V_\ell$ each of size $2n/k$. In $V_0$, we place $\mathrm{BE}(|V_0|)$, and, for each $i \in [\ell]$, we place a copy of the $\mathrm{ER}(|V_i|)$ on $V_i$, the minimum degree of this graph is $\left(\frac{k-3}{k} + \frac{1}{4} \cdot \frac{3}{k} - o(1)\right) n = \left(\frac{4k-9}{4k} - o(1)\right) n$.

It should also be noted that when $\alpha(G)$ is at most a constant, the fact that $G$ has a $K_k$-tiling on all but at most a constant number of vertices is a direct consequence of Ramsey's Theorem. Furthermore, when we add the condition $\delta(G) \geq (1/2 + \varepsilon)n$, a counting argument and Ramsey's Theorem show that there are $\Omega(n^{k-1})$ copies of $K_{k-1}$ in the intersection of the neighborhoods of any two distinct vertices, so the absorbing method gives a $K_k$-factor.

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[2] Strictly speaking, for any fixed $n$, both $\mathrm{ER}(n)$ and $\mathrm{BE}(n)$ are families of graphs.

# Appendix (by Christian Reiher and Mathias Schacht): Clique factors in locally dense graphs

Balogh, Molla, and Sharifzadeh study sufficient minimum degree conditions for $K_k$-factors in graphs with sublinear independence number. In particular, for $K_k$-factors the minimum degree condition for $n$-vertex graphs is of the form $c_k n$ where $c_k \to 1$ as $k \to \infty$.

In this appendix we note that if we require positive density on all linear sized subsets, instead of just one edge, then for $n$-vertex graphs the minimum degree condition

$$\delta(G) \geq (1/2 + o(1))n$$

suffices for $K_k$-factors for any $k \geq 3$. More formally, we say a graph $G = (V, E)$ is $(\varrho, d)$-*dense*, if for every $U \subseteq V$ the number $e_G(U)$ of edges of $G$ induced on $U$ is at least $d\binom{|U|}{2} - \varrho|V|^2$. In the proof of Theorem A.1 below we shall utilise the well-known fact, that for any $k$, $d$, and $\xi > 0$, there exists $\varrho > 0$ such that every sufficiently large $(\varrho, d)$-dense graph $G = (V, E)$ contains at least $(d^{\binom{k}{2}} - \xi)|V|^k$ labeled cliques $K_k$.

**Theorem A.1.** *For every integer $k \geq 3$, $\varepsilon > 0$, and $d > 0$ there exist $\varrho > 0$ and $m_0$ such that for every integer $m \geq m_0$ the following holds: If $G = (V, E)$ is a $(\varrho, d)$-dense graph on $|V| = n = km$ vertices with $\delta(G) \geq (1/2 + \varepsilon)n$, then $G$ contains a $K_k$-factor.*

*Proof (Sketch).* The proof is based on the absorption method of Rödl, Ruciński, and Szemerédi introduced in [28]. We will fix some auxiliary constants $d'$, $d''$, and $\eta$ in such a way that the following hierarchy is imposed

$$\tfrac{1}{k},\, d,\, \varepsilon \gg d' \gg d'' \gg \eta \gg \varrho\,.$$

It is easy to see that those constants can be chosen in such a way that in any sufficiently large $(\varrho, d)$-dense graph $G = (V, E)$ one can remove the vertex sets of copies of $K_k$'s in a greedy manner until only $\eta|V|$ vertices are left. In other words, this observation reduces the proof of Theorem A.1 to the problem to ensure the abundant existence of suitable absorbers in $G$. Here we may use the minimum degree condition, which allows us to apply the $(\varrho, d)$-denseness condition within the joint neighbourhoods of any pair of vertices. This will allow us to find absorbers, which are very similar to those appearing in [16].

Given distinct vertices $v_1, \dots, v_k \in V$ we observe that the subgraph $G[N(v_1)]$ induced on the neighbourhood of $v_1$ is still $(4\varrho, d)$-dense and, hence, there exist $d'n^{k-1}$ cliques $K_{k-1}$ that extends $v_1$ to a $K_k$. Let $u_2, \dots, u_k$ be such a clique disjoint from $v_1, \dots, v_k$. For $j = 2, \dots, k$ we consider the joint neighbourhood $N(v_j, u_j) = N(v_j) \cap N(u_j)$. Owing to the minimum degree condition we have $|N(v_j, u_j)| \geq 2\varepsilon n$. Therefore, $G[N(v_j, u_j)]$ is $(\frac{\varrho}{4\varepsilon^2}, d)$-dense and, hence, there are $\Omega(n^{k-1})$ cliques $K_{k-1}$ in the joint neighbourhood of $u_j$ and $v_j$.

Summarising, we have shown that for any given distinct vertices $v_1, \dots, v_k \in V$ there exist $d''n^{k(k-1)}$ collections of disjoint cliques $K^1, K^2, \dots, K^k$ of order $k-1$ with $V(K^1) = \{u_2, \dots, u_k\}$ such that $v_1 + K^1, v_2 + K^2, \dots, v_k + K^k$ and $u_2 + K^2, \dots, u_k + K^k$ form copies of $K_k$ in $G$. In particular, $u_2 + K^2, \dots, u_k + K^k$ form a $K_k$-factor on

$$V(K^1) \cup V(K^2) \cup \dots \cup V(K^k)$$

and $v_1 + K^1, v_2 + K^2, \ldots, v_k + K^k$ form a $K_k$-factor on

$$V(K^1) \cup V(K^2) \cup \cdots \cup V(K^k) \cup \{v_1, \ldots, v_k\}\,.$$

In other words, such a collection $K^1, K^2, \ldots, K^k$ forms an absorber for the given vertices $v_1, \ldots, v_k$ and for any given $v_1, \ldots, v_k$ there are at least $d'' n^{k(k-1)}$ such absorbers. The theorem then follows by a standard application of the absorption method and we omit the details. □

The degree condition of Theorem A.1 is approximately best possible, as the example following Theorem 1.2 shows. It seems plausible that constructions of this kind (two cliques of order roughly $n/2$ that share up to $k-2$ vertices) lead to optimal lower bounds for the minimum degree condition. Therefore, we put forward the following question.

**Question A.2.** *Is it true that $\delta(G) \geq n/2 + O(1)$ suffices in Theorem A.1?*

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