Suppose we are given the problem

Minimize \( z = -19x_1 - 13x_2 - 12x_3 - 17x_4 \)

subject to

\[
\begin{align*}
3x_1 + 2x_2 + x_3 + 2x_4 &= 225, \\
x_1 + x_2 + x_3 + x_4 &= 117, \\
4x_1 + 3x_2 + 3x_3 + 4x_4 &= 420, \\
x_1, x_2, x_3, x_4 &\geq 0.
\end{align*}
\] (1)

- There is no obvious bfs, so we use the revised two phase simplex method
- To start the first phase, we add to each of the equations its own variable \( y_i \) and consider the auxiliary problem of minimizing \( \zeta = y_1 + y_2 + y_3 \) (we think of \( y_1 = x_5 \), \( y_2 = x_6 \) and \( y_3 = x_7 \))
- Throughout the first phase, \( c^T \) and \( A \) refer to the cost vector and matrix of the first phase linear program, not the original LP (1).
- In the second phase, \( c^T \) and \( A \) refer to the cost vector and matrix of the original LP (1).

This is the tableau corresponding to the phase one LP

<table>
<thead>
<tr>
<th>( -\zeta )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>225</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>117</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( y_4 )</td>
<td>420</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Since \( b \geq 0 \), row one is solved \( y_1 \), row two is solved \( y_2 \), and row three is solved \( y_3 \), we can use \((5, 6, 7)\) as our ordered basis.

- Note that we do not exclude \( y_1, y_2 \) and \( y_3 \) from the top row.
- Our carry matrix should have the form \( \begin{bmatrix} -\pi^T b & -\pi^T \\ A_0^{-1} b & A_0^{-1} \end{bmatrix} \).
- Note that \( A_0^{-1} = A_0 \) is the identity matrix. We compute \( \pi^T = c_0^T A_0 = c_0^T = [1, 1, 1] \) and \( A_0^{-1} b = \{225, 117, 420\}^T \).
- We have \( \pi^T b = [1, 1, 1][225, 117, 420]^T = 225 + 117 + 420 = 762 \).
- The following is then our CARRY-0 matrix

<table>
<thead>
<tr>
<th>( -\zeta )</th>
<th>( -762 )</th>
<th>(-1)</th>
<th>(-1)</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>225</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>117</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>420</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

We compute \( C_1 = c_1 - \pi^T A_1 = 0 + [-1, -1, -1][3, 1, 4]^T = -8 < 0 \), so we pivot on column 1.

We compute \( A_0^{-1} A_1 = A_1 = [3, 1, 4]^T \).

So, we append column \([-8, 3, 1, 4]^T \) to CARRY-0 and pivot. We do the normal ratio test to the select the pivot row, i.e. we pick row one as the pivot row since \( 225/3 < 117/1 \) and \( 225/3 < 420/4 \).

After pivoting, we get CARRY-1

<table>
<thead>
<tr>
<th>( -\zeta )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>75</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>42</td>
<td>-1/3</td>
<td>1</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>120</td>
<td>-4/3</td>
<td>0</td>
</tr>
</tbody>
</table>
Now we compute $\pi_2 = c_2 - \pi^T A_2 = 0 + [5/3, -1, -1][2, 1, 3]^T = -2/3$ (we do not calculate $\pi_1$, because $x_1$ is in the basis, so we know $\pi_1 = 0$.)

Since $\pi_2 = -2/3 < 0$, we pivot on column 2.

We compute $A_B^{-1} A_2 = \begin{pmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ -4/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}.$

Adding column $[-2/3, 2/3, 1/3, 1/3]^T$ to CARRY-1 and pivoting on the first row we get CARRY-2:

<table>
<thead>
<tr>
<th>-ξ</th>
<th>-69</th>
<th>0</th>
<th>3</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₂</td>
<td>108</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>x₃</td>
<td>9</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>y₃</td>
<td>69</td>
<td>0</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

Since $x₂$ is in the basis and $x₃$ was removed from the basis on the previous step, we can start with column 3. (On the next iteration, we will have to check the $x₃$ column again.)

We compute $\pi_3 = c_3 - \pi^T A_3 = 0 + [2, -1, -1][1, 1, 3]^T = -2 < 0$, so we pivot on column 3.

Now we compute $A_B^{-1} A_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 3/2 \end{pmatrix}.$

Adding column $[-2, 1/2, 1/2, 3/2]^T$ to CARRY-2 and pivoting on the second row we get CARRY-3:

<table>
<thead>
<tr>
<th>-ξ</th>
<th>-69</th>
<th>0</th>
<th>3</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₂</td>
<td>108</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>x₃</td>
<td>9</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>y₃</td>
<td>69</td>
<td>0</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

We know must check column 1 again, so we compute $\pi_1 = c_1 - \pi^T A_1 = 0 + [0, 3, -1][3, 1, 4]^T = -1 < 0$, so we pivot on column 1.

$A_B^{-1} A_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$

We add column $[-1, 2, -1, 1]^T$ to CARRY-3 and pivot on the first row to get CARRY-4:

<table>
<thead>
<tr>
<th>-ξ</th>
<th>-15</th>
<th>1/2</th>
<th>5/2</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₁</td>
<td>54</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
</tr>
<tr>
<td>x₃</td>
<td>63</td>
<td>-1/2</td>
<td>3/2</td>
<td>0</td>
</tr>
<tr>
<td>y₃</td>
<td>15</td>
<td>-1/2</td>
<td>-5/2</td>
<td>1</td>
</tr>
</tbody>
</table>
Hence, our CARRY-6 is
\[
\begin{array}{ccc}
-\zeta & 1827 & 2 1 3 \\
x_1 & 39 & 1 2 -1 \\
x_3 & 48 & 0 4 -1 \\
x_4 & 30 & -1 -5 2
\end{array}
\]

Since \( x_3 \) entered the basis, then left it, and now entered it again.

Since \( x_1 \) and \( x_3 \) are in the basis and \( x_2 \) was just removed from the basis on the last iteration, we can start with column 4:
\[
\zeta_4 = c_4 - \pi^T A_4 = 0 + \begin{bmatrix} 1/2 & 5/2 & -1 \end{bmatrix}^T = -1/2 < 0.
\]
So we pivot on column 4,
\[
A^{-1}_B A_4 = \begin{pmatrix}
1/2 & -1/2 & 0 \\
-1/2 & 3/2 & 0 \\
-1/2 & -5/2 & 1
\end{pmatrix}
\begin{pmatrix}
2 \\
1 \\
4
\end{pmatrix}
= \begin{pmatrix}
2/3 \\
1/3 \\
1/3
\end{pmatrix}.
\]

Adding column \([-1/2, 1/2, 1/2, 1/2]^T\) to the last tableau and pivoting on the last row we get CARRY-5:
\[
\begin{array}{ccccc}
-\zeta & 0 & 0 & 0 & 0 \\
x_1 & 39 & 1 & 2 & -1 \\
x_3 & 48 & 0 & 4 & -1 \\
x_4 & 30 & -1 & -5 & 2
\end{array}
\]

Since \( \zeta = 0 \) and \( y_1, y_2 \) and \( y_3 \) are not in the basis, we have found a feasible ordered basis for the original problem \( B = (1, 3, 4) \).

We replace the top row with \([-\pi^T b - \pi^T\]) where \( \pi^T = c_B^T A_B^{-1} \) is computed using \( c^T \) from the original LP (1).

We compute (note the order \( c_B^T = [c_1, c_2, c_4] = [-19, -12, -17] \) must match the order in the basis heading \( x_1, x_2, x_3, x_4 \))
\[
\pi^T = c_B^T A_B^{-1} = [-19, -12, -17] \begin{pmatrix}
1 & 2 & -1 \\
0 & 4 & -1 \\
-1 & -5 & 2
\end{pmatrix} = [-2, -1, -3].
\]

Then we compute \( \pi^T b = [-2, -1, -3][255, 117, 420]^T = -1827 \).

Hence, our CARRY-6 is
\[
\begin{array}{ccc}
-z & 1827 & 2 1 3 \\
x_1 & 39 & 1 2 -1 \\
x_3 & 48 & 0 4 -1 \\
x_4 & 30 & -1 -5 2
\end{array}
\]

The only variable not in the basis is \( x_2 \), so we compute
\[
\zeta_2 = c_2 - \pi^T A_2 = -13 + \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T = 1 \geq 0,
\]
Since it is not negative, we conclude that the optimal value is \(-1827\) attained at \( x = [39, 0, 48, 30]^T \).