Let $G = (V,E)$ be a directed graph with cost function $c : E \to \mathbb{R}^+$ on the edges. We let $m := |E|$ be the number of edges of $G$ and $n := |V|$ be the number of vertices of $G$. We wish to find a shortest or min-cost path from vertex $s$ to vertex $t$. We let $A$ be the incidence matrix for $G$ with the row corresponding to $t$ removed and we attempt to find a solution to the following LP:

$$(P) \quad \text{min } c^T f \text{ such that } Af = d, f \geq 0.$$ 

Here $d$ is defined by

$$d_i = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases},$$

so we essentially want one unit of flow to leave $s$, and flow to be conversed at every other vertex except $t$. This implies that one unit of flow will enter $t$. We can remove the last row (or any row) of $A$, because $A$ does not have full rank since the sum of the rows of an incident matrix of any directed graph is 0.

The dual is

$$(D) \quad \text{max } \pi_s \text{ such that } \pi_i - \pi_j = c_{ij} \forall (i,j) \in E, \pi_t = 0.$$ 

Notice how we have associated the variables in the dual with the vertices in of $G$. Here we add the extra variable $\pi_t = 0$ and set it to 0, this is necessary because there is not a row corresponding to $t$ in the original primal LP.

**Exercise:** If $\pi$ is feasible for $(D)$, prove that for any $v \in V$, $\pi_v$ is at most the cost of a shortest path from $v$ to $t$.

To start the primal dual process, assume the $\pi$ is feasible for $D$, $(\pi = 0$ is feasible if $c \geq 0$) and let $J = \{(i,j) \in E : \pi_i - \pi_j = c_{ij}\}$. Call a path in $G$ a $J$-path if it consists entirely of edges from the set $J$. Let

$$W = \{v \in V : \text{ there is a } J\text{-path from } x \text{ to } t \text{ in } G \}$$

and let $\overline{W} = V \setminus W$.

**Exercise:** If $\pi$ is feasible for $(D)$, prove that for any $v \in W$, $\pi_v$ is exactly the cost of a shortest path from $v$ to $t$.

Now we can write the restricted primal program, let $x_r \in \mathbb{R}^m$ and $\hat{f} = [x^*|f_J]^T$.

$$\text{(RP) } \text{min } \xi = \begin{bmatrix} 1^T & 0^T \end{bmatrix} \hat{f} \text{ such that } [I_m|A_J] \hat{f} = d, \hat{f} \geq 0.$$ 

Clearly (RP) is feasible and bounded, so we can let $\xi_{opt}$ be the optimal value of (RP). The following is the dual of the restricted primal

$$(\text{DRP}) \quad \text{max } \pi_s \text{ such that } \begin{array} \pi_i - \pi_j & \leq 0 & \text{ for all } (i,j) \in J \\ \pi_i - \pi_j & \leq 1 & \text{ for all } (i,j) \in E \\ \pi_t & = 0. \end{array}$$

Assume that $\pi$ is feasible for (DRP). Note that if $\pi$ is optimal and $\pi_s = 0$, then $\xi_{opt} = 0$ and we are done. Also, note that if $\pi$ is feasible for (DRP) and $\pi_s = 1$, then $\pi$ is clearly optimal.
Let \( x \in W \). Since \( x \in W \), there exists a \( J \)-path \( x = v_1v_w \cdots v_{d-1}v_d = t \) from \( x \) to \( t \). Since \( \pi \) is feasible we have that inequalities
\[
0 = \pi_t = \pi_{v_d} \geq \pi_{v_{d-1}} \geq \cdots \geq \pi_{v_1} = \pi_x,
\]
so \( \pi_x \leq 0 \). Since \( \pi = 0 \) is feasible for (DRP), when \( s \in W, \pi = 0 \) is optimal and we are done.

Let
\[
\pi_i = \begin{cases} 
0 & \text{if } i \in W \\
1 & \text{if } i \in \overline{W}.
\end{cases}
\]

We claim that \( \pi \) is feasible for DRP. To see this note that for every \((i, j) \in E, \pi_i - \pi_j \leq 0\), unless \( i \in \overline{W} \) and \( j \in W \) and \( \pi_i - \pi_j = 1 \), but in this case, by the definition of \( W \), \((i, j) \notin J \).

We compute
\[
\theta = \min_{(i,j) \in J} \left\{ \frac{c_{ij} - (\pi_i - \pi_j)}{\pi_i - \pi_j} \right\} = \min_{(i,j) \in E \text{ and } i \in \overline{W} \text{ and } j \in W} \{c_{ij} - (\pi_i - \pi_j)\}.
\]

So we now have our the following algorithm:

1. Start with \( \pi = 0 \) and let \( W = \{t\} \) and \( \overline{W} = V \setminus \{t\} \).
2. Compute \( \theta = \min_{(i,j) \in E \text{ and } i \in \overline{W} \text{ and } j \in W} \{c_{ij} - (\pi_i - \pi_j)\} \).
3. Add \( \theta \) to \( \pi_i \) if \( i \in \overline{W} \).
4. Add \( i \) to \( W \) and remove it from \( \overline{W} \), if there exists an edge \((i, j) \in E \) such that \( j \in \overline{W} \) and \( c_{ij} - (\pi_i - \pi_j) = \theta \). Note that there will always exist at least one such vertex \( i \).
5. If \( s \in W \) we are done, \( \pi_s \) is the length of the shortest path from \( s \) to \( t \). Otherwise, repeat from step 2.

**Exercise:** Prove that once \( x \in W \) it is in \( W \) on every iteration.