An example of the primal–dual simplex method

Suppose we are given the problem $P$:

Maximize $z = -x_1 - 3x_2 - 3x_3 - x_4$

subject to

\[
\begin{align*}
3x_1 + 4x_2 - 3x_3 + x_4 &= 2, \\
3x_1 - 2x_2 + 6x_3 - x_4 &= 1, \\
6x_1 + 4x_2 + x_4 &= 4, \\
x_1, x_2, x_3, x_4 &\geq 0.
\end{align*}
\] (1)

The dual to $P$ is of course the following $D$

Minimize $w = 2\pi_1 + \pi_2 + 4\pi_3$

subject to

\[
\begin{align*}
3\pi_1 + 3\pi_2 + 6\pi_3 &\geq -1, \\
4\pi_1 - 2\pi_2 + 4\pi_3 &\geq -3, \\
-3\pi_1 + 6\pi_2 &\geq -3, \\
\pi_1 - \pi_2 + \pi_3 &\geq -1.
\end{align*}
\] (2)

Somebody tells us that probably vector $\pi = (-1/3, 0, 0)^T$ is an optimal vector in $D$. Note that the value of $w$ with this $\pi$ is $-2/3$. We start checking this version using complementary slackness. First, we plug this vector in $D$ and see that it is a feasible vector and only the first inequality is binding. Hence our first set $J$ is $\{1\}$. In particular, if $\pi$ is an optimal vector in $D$, then in the corresponding optimal vector $x$ of $P$ only coordinate $x_1$ can be non-zero. We try to find it by solving the following restricted primal problem $RP1$:

Maximize $\xi = -x_1^r - x_2^r - x_3^r$

subject to

\[
\begin{align*}
3x_1 + x_1^r &= 2, \\
3x_1 + x_2^r &= 1, \\
6x_1 + x_3^r &= 4, \\
x_1, x_1^r, x_2^r, x_3^r &\geq 0.
\end{align*}
\] (3)

Normally, we would use the revised simplex to solve it. But here we will write down all the tableaus. So, the initial tableau is

<table>
<thead>
<tr>
<th>y_0 = -\xi</th>
<th>x_1</th>
<th>x_1^r</th>
<th>x_2^r</th>
<th>x_3^r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Excluding $x_1^r, x_2^r, \text{and } x_3^r$ from Row 0, we have

<table>
<thead>
<tr>
<th>y_0 = -\xi</th>
<th>x_1</th>
<th>x_1^r</th>
<th>x_2^r</th>
<th>x_3^r</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
We pivot on $a_{2,1}$ and get

$$y_0 = -\xi$$

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_1^r$</th>
<th>$x_2$</th>
<th>$x_3^r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>1/3</td>
<td>1</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

This is the final tableau which proves that our $\pi = (-1/3, 0, 0)^T$ is NOT optimal. But this is not only a negative outcome, since we now know how to improve the $\pi$. Our new $\pi^*$ will have the form

$$\pi^* = \pi + \theta \pi,$$

where $\theta$ is a positive factor that we will find below and $\pi$ is an optimal vector in the dual DRP1 to RP1 which (by definition) is as follows:

$$\text{Minimize } w^r = 2\pi_1 + \pi_2 + 4\pi_3$$

subject to

$$\begin{align*}
3\pi_1 &+ 3\pi_2 + 6\pi_3 \geq 0, \\
\pi_1 &\geq -1, \\
\pi_2 &\geq -1, \\
\pi_3 &\geq -1.
\end{align*}$$

We can find $\pi$, from the last tableau for RP1, where the vector $(0, -4, 0)$ in Row 0 is in fact $(-1, -1, -1) - (\pi_1, \pi_2, \pi_3)$. Hence $(\pi_1, \pi_2, \pi_3) = (-1, -1, -1) - (0, -4, 0) = (-1, 3, -1)$. Now we choose the maximum $\theta$ such that the vector $(\pi^*)^T = (-1/3, 0, 0) + \theta(-1, 3, -1)$ is feasible in $D$. Plugging this $\pi^*$ into the first inequality of $D$ we get the inequality

$$(\pi^*)^T \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = (-1/3, 0, 0) \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} + \theta(-1, 3, -1) \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = -1 + \theta \cdot 0 = -1 \geq -1,$$

which holds for every $\theta$. Similarly, plugging $\pi^*$ into the second inequality of $D$ we get the inequality $-4/3 - \theta 14 \geq -3$ which holds for $\theta \leq 5/42$. Plugging $\pi^*$ into the third inequality of $D$ we get $1 + \theta (21) \geq -3$ which holds for every positive $\theta$. Finally, plugging $\pi^*$ into the fourth inequality of $D$ we get $-1/3 + \theta \cdot 5 \geq -1$ which holds for $\theta \leq 2/15$. Thus we choose $\theta = 5/42$ and hence our new $\pi = \pi^* \text{ is } (-1/3, 0, 0)^T + 5/42(-1, 3, -1)^T = (-19/42, 5/14, -5/42)^T$. Note that now $w = 2-19/42 + -5/14 + 4-5/42 = -43/42$.

So, we start our cycle again. We hope that the new $\pi$ is optimal. Plugging it in $D$ we see that now $J = \{1, 2\}$. Thus, our new restricted primal RP2 is

$$\text{Maximize } \xi = -x_1^r - x_2^r - x_3^r$$

subject to

$$\begin{align*}
3x_1 &+ 4x_2 &+ x_1^r &\quad = 2, \\
3x_1 &- 2x_2 &+ x_2^r &\quad = 1, \\
6x_1 &+ 4x_2 &+ x_3^r &\quad = 4, \\
x_1, & \quad x_2, & \quad x_1^r, & \quad x_2^r, & \quad x_3^r &\quad \geq 0.
\end{align*}$$

(5)
But we do not start from scratch. We use the last tableau of the previous iteration adding there the values of the $x_2$-column obtained from knowing $A_B^{-1}$:

$$y_0 = -\xi$$

$$\begin{array}{cccccc}
    x_1 & x_2 & x_1^r & x_2^r & x_3^r \\
    3 & 0 & 14 & 0 & -4 & 0 \\
    1 & 0 & 6 & 1 & -1 & 0 \\
    1/3 & 1 & -2/3 & 0 & 1/3 & 0 \\
    2 & 0 & 8 & 0 & -2 & 1 \\
\end{array}$$

Here, the second column was obtained using the formulas $\tilde{c}_2 = c_2 - (-1, 3, -1) \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = 0 + 14 = 14$, and $\tilde{A}_2 = A_B^{-1}A_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/3 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -2/3 \\ 8 \end{pmatrix}$. Note that $A_B^{-1}$ is in the last three rows and columns of the previous tableau.

We pivot on $a_{1,2}$ and get

$$y_0 = -\xi$$

$$\begin{array}{cccccc}
    x_1 & x_2 & x_1^r & x_2^r & x_3^r \\
    2/3 & 0 & 0 & -7/3 & -5/3 & 0 \\
    1/6 & 0 & 1 & 1/6 & -1/6 & 0 \\
    4/9 & 1 & 0 & 1/9 & 2/9 & 0 \\
    2/3 & 0 & 0 & -4/3 & -2/3 & 1 \\
\end{array}$$

This is the final tableau which proves that our new $\pi$ again is not optimal. So, we again correct it using (4). Recall that our restricted dual D\textsc{RP2} is

$$\begin{align*}
\text{Minimize } & w^r = 2\pi_1 + \pi_2 + 4\pi_3 \\
\text{subject to } & \begin{cases} 
3\pi_1 + 3\pi_2 + 6\pi_3 \geq 0, \\
4\pi_1 - 2\pi_2 + 4\pi_3 \geq 0, \\
\pi_1 \geq -1, \\
\pi_2 \geq -1, \\
\pi_3 \geq -1.
\end{cases}
\end{align*}$$

Similarly to the previous iteration, we have $(\pi_1, \pi_2, \pi_3) = (-1, -1, -1) - (-7/3, -5/3, 0) = (4/3, 2/3, -1)$. To find the maximum $\theta$ such that the vector $\pi^* = (-19/32, 5/14, -5/12)^T + \theta(4/3, 2/3, -1)^T$ is feasible in $D$, we plug this $\pi^*$ into all inequalities of $D$. From the first inequality we get

$$(\pi^*)^T \begin{pmatrix} 3 \\ 3 \\ 6 \\
\end{pmatrix} = \left(-19/42, 5/14, -5/42 \right) \begin{pmatrix} 3 \\ 3 \\ 6 \\
\end{pmatrix} + \theta(4/3, 2/3, -1) \begin{pmatrix} 3 \\ 3 \\ 6 \\
\end{pmatrix} = -1 + \theta(4 + 2 - 6) = -1 \geq -1,$$

which holds for every $\theta$. Similarly, from the second inequality of $D$ we get $-3 + \theta(16/3 - 4/3 - 4) \geq -3$ which also holds for every $\theta$. From the third inequality of $D$ we get $7/2 + \theta(-4 + 4 + 0) \geq -3$ which holds for every $\theta$. Finally, from the fourth inequality of $D$ we get $-13/14 + \theta(4/3 - 2/3 - 1) \geq -1$ which holds for $\theta \leq 3/14$. 

3
Thus we choose $\theta = 3/4$ and hence our new $\pi$ is $(-\frac{1}{6}, \frac{5}{18}, \frac{5}{18})^T + \frac{3}{14}(4/3, 2/3, -1)^T = (-\frac{1}{6}, \frac{1}{2}, -\frac{1}{4})^T$. Note that now $w = 2\frac{1}{6} + \frac{1}{2} + 4\frac{1}{4} = \frac{7}{6}$.

We start our cycle again. Now $J = \{1, 2, 4\}$. Thus, our new restricted primal RP3 is

\[
\text{Maximize } \xi = x_1^r + x_2^r + x_3^r
\]

subject to

\[
\begin{align*}
3x_1 + 4x_2 + x_4 + x_1^r &= 2, \\
3x_1 - 2x_2 - x_4 + x_2^r &= 1, \\
6x_1 + 4x_2 + x_4 + x_3^r &= 4,
\end{align*}
\]

\[x_1, x_2, x_4, x_1^r, x_2^r, x_3^r \geq 0.\]

We use the modified last tableau

\[
y_0 = -\xi \quad \begin{array}{cccccc} 
2/3 & 0 & 0 & 1/3 & -7/3 & -5/3 & 0 \\
1/6 & 0 & 1 & 1/3 & 1/6 & -1/6 & 0 \\
4/9 & 1 & 0 & -1/9 & 1/9 & 2/9 & 0 \\
2/3 & 0 & 0 & 1/3 & -4/3 & -2/3 & 1 \\
\end{array}
\]

where the third column was obtained using the formulas

\[
\bar{c}_4 = c_4 - (4/3, 2/3, -1) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 + 1/3 = 1/3,
\]

and $\bar{A}_4 = A_B^{-1}A_4 = \begin{pmatrix} 1/6 & -1/6 & 0 \\ 1/9 & 2/9 & 0 \\ -4/3 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/9 \\ 1/3 \end{pmatrix}$.

We pivot on $x_4$-column and Row 1. The result is

\[
y_0 = -\xi \quad \begin{array}{cccccc} 
1/2 & 0 & -1 & 0 & -5/2 & -3/2 & 0 \\
1/2 & 0 & 3 & 1 & 1/2 & -1/2 & 0 \\
1/2 & 1 & 1/3 & 0 & 1/6 & 1/6 & 0 \\
1/2 & 0 & -1 & 0 & -3/2 & -1/2 & 1 \\
\end{array}
\]

As above, the optimal vector of the new restricted dual DRP3 is $(\pi_1, \pi_2, \pi_3)^T = (-1, -1, -1)^T - (5/2, -3/2, 0)^T = (3/2, 1/2, -1)^T$. To find the maximum $\theta$ such that the vector $\pi^* = (-\frac{1}{6}, \frac{1}{2}, -\frac{1}{3})^T + \theta(3/2, 1/2, -1)^T$ is feasible in $D$, we do not need to check inequalities in $D$ corresponding to $x_1$ and $x_4$, since they are in the basis of RP3. From the remaining two inequalities we get

\[
(\pi^*)^T \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = (\frac{1}{6}, \frac{1}{2}, -\frac{1}{3}) \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} + \theta(3/2, -1/2, -1) \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix} = 7/2+\theta \cdot (-9/2+3+0) \geq -3,
\]
which holds for $\theta \leq 13/3$, and $-3 + \theta(6 - 1 - 4) \geq -3$ which holds for each positive $\theta$.

Thus we choose $\theta = 13/3$ and hence our new $\pi$ is $(-1/6, 1/3, -1/3)^T + 13/3(3/2, 1/2, -1)^T = (19/3, 8/3, -14/3)^T$. Note that now $w = -2 - 19/3 + 8/3 - 44/3 = -19/3$.

We start our cycle again. Now $J = \{1, 3, 4\}$. Note that 2 is not in $J$ anymore.

The tableau corresponding to the new restricted primal $\text{RP4}$ is

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_1^r$</th>
<th>$x_2^r$</th>
<th>$x_3^r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_0$</td>
<td>$-\xi$</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>-5/2</td>
<td>-3/2</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1/2</td>
<td>0</td>
<td>-9/2</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1/2</td>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>1/6</td>
<td>1/6</td>
</tr>
<tr>
<td>$x_3^r$</td>
<td>1/2</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>-3/2</td>
<td>-1/2</td>
</tr>
</tbody>
</table>

We got the column for $x_3$ using the formulas

$$\tilde{c}_3 = c_3 - (3/2, 1/2, -1)^T \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = 0 + 9/2 - 6/2 = 3/2,$$

and $\tilde{A}_3 = A_B^{-1}A_3 = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/6 & 1/6 & 0 \\ -3/2 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -9/2 \\ 1/2 \\ 3/2 \end{pmatrix}$.

Pivoting on $x_3$-column and Row 3 we get

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_1^r$</th>
<th>$x_2^r$</th>
<th>$x_3^r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_0$</td>
<td>$-\xi$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-4</td>
<td>-2</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1/3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2/3</td>
<td>1/3</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1/3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1/3</td>
</tr>
</tbody>
</table>

So, vector $(19/3, 8/3, -14/3)^T$ indeed is an optimal vector in $D$ and the corresponding optimal vector in $P$ is $(1/3, 0, 1/3, 2)^T$. The optimal cost is $-10/3$. 
