Traveling Salesman Problem

Math 482, Lecture 35

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Important special case: cities are points in the plane, and c_{ij} is the distance from i to j.

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Leave each city exactly once:

$$\sum_{\substack{1 \le k \le n \\ k \ne j}} x_{jk} = 1 \qquad \text{for each } j = 1, 2, \ldots, n.$$

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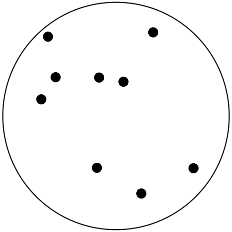
Here is a TSP instance with 9 cities; assume that cost is Euclidean distance.



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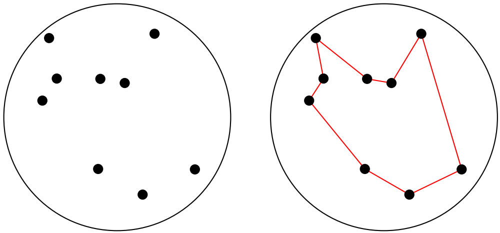

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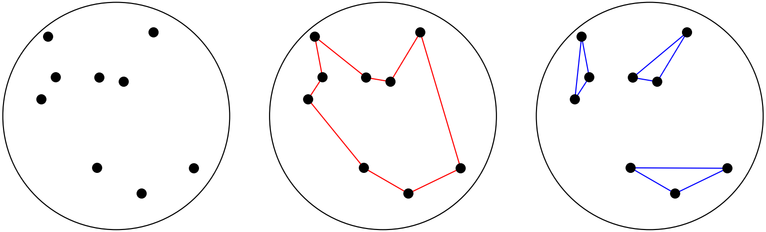

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The optimal solution to the local constraints is in blue. It has three *subtours* that are not connected to each other.

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In a solution to the local constraints with subtours, this is violated if we take S to be the set of cities in a subtour.

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For example, let S be the set of all cities visited by the tour, starting at city 1.

- If $S = \{1, 2, \ldots, n\}$, we actually do have a tour.
- Otherwise, the constraint saying we must leave S at least once is violated.

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3. If it's an integer solution representing a tour, update our best solution found!

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We want an inequality to encode the logical implication

$$\text{if } x_{ij} = 1, \text{ then } t_j \geq t_i + 1$$

for every pair of cities $i, j \neq 1$.

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   If we visit cities $i_1, i_2, \ldots, i_{n-1}$ in that order from city 1, set $t_{i_1} = 1, t_{i_2} = 2, \ldots, t_{i_{n-1}} = n - 1$.

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   Suppose $x_{ab} = x_{bc} = x_{ca} = 1$ and none of a, b, c are 1.

   Then we can't satisfy the three constraints

$$
\begin{aligned}
t_b &\geq t_a + 1 \\
t_c &\geq t_b + 1 \\
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We can check: if we take $M = n$, then any actual tour can satisfy these constraints. The times $t_2, \dots, t_n$ can be chosen between 1 and $n - 1$, so $t_j \geq t_i + 1 - n$ always holds.

Comparing the methods

At first glance:

- DFJ's formulation has $2^{n-1} - 1$ extra constraints, plus the $2n$ local constraints.
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In practice:

- DFJ's formulation has an efficient branch-and-cut approach.
- MTZ's formulation is weaker: the feasible region has the same integer points, but includes more fractional points.