Branch-and-Bound

Math 482, Lecture 33

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- When a subproblem is too hard to solve directly, we at least put a **bound** on its objective value to let us eliminate branches without having to look at all of them.

For example: if subproblem A definitely achieves an objective value of 100 (and we're maximizing), and subproblem B's objective value is at most 80, we can **prune** subproblem B without breaking it down into further cases.

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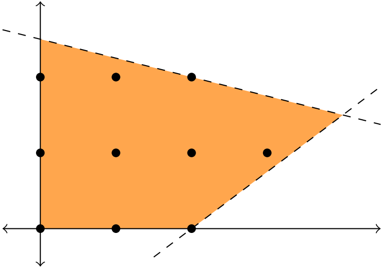
- We can bound the value of an integer program by solving its **linear relaxation**: the LP where we forget about the integer constraints.
- A subproblem is "easy" if the linear relaxation happens to have an integer solution. Otherwise, we will need to branch on it.
- To branch on a fractional solution where $x_i = f \notin \mathbb{Z}$, take the following two subproblems:
  - one where we add the constraint $x_i \le \lfloor f \rfloor$, and
  - one where we add the constraint $x_i \ge \lceil f \rceil$.

Branch-and-bound example

We will use branch and bound to solve the following linear program:

$$\begin{aligned}
\underset{x,y\in\mathbb{Z}}{\text{maximize}} \quad & 4x + 5y \\
\text{subject to} \quad & x + 4y \leq 10 \\
& 3x - 4y \leq 6 \\
& x, y \geq 0
\end{aligned}$$



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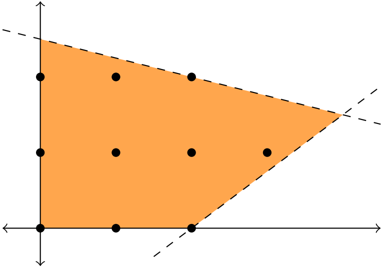

Step 1: solve the LP relaxation. This has optimal solution $(x, y) = (4, 1.5)$ with $4x + 5y = 23.5$.

The branch step, geometrically

Since the optimal solution has $y = 1.5 \notin Z$, we can consider two cases that both eliminate this point: $y \leq 1$, or $y \geq 2$.

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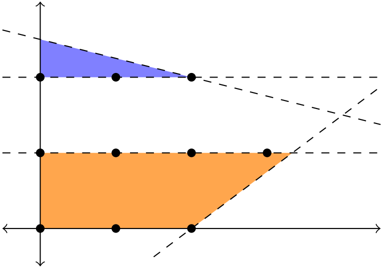

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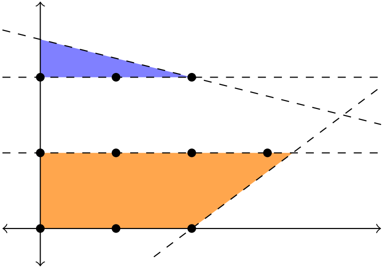

(Note: we must get rid of the point $(4, 1.5)$ in future cases we consider, or we'll just get it back as the optimal solution again!)

The branch step in the simplex tableau

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3. Row-reduce the resulting tableau.

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The general method:

1. Take the optimal simplex tableau for the previous subproblem.
2. Add a new row (and slack variable) for the new constraint we add.
3. Row-reduce the resulting tableau.
4. Solve with the dual simplex method.

The branch step: an example

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Step 1: take the optimal tableau

	x	y	s_1	s_2	
y	0	1	$3/16$	$-1/16$	$3/2$
x	1	0	$1/4$	$1/4$	4
$-z$	0	0	$-31/16$	$-11/16$	$-47/2$

The branch step: an example

Here's how we do this to add a $y \geq 2$ constraint to the LP that gave us $(x, y) = (4, 1.5)$.

Step 2: Add a new row for "$-y + s_3 = -2$"

	x	y	s_1	s_2	s_3	
y	0	1	$3/16$	$-1/16$	0	$3/2$
x	1	0	$1/4$	$1/4$	0	4
s_3	0	-1	0	0	1	-2
$-z$	0	0	$-31/16$	$-11/16$	0	$-47/2$

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Step 3: Row-reduce this tableau

	x	y	s_1	s_2	s_3	
y	0	1	$^3/_{16}$	$-^1/_{16}$	0	$^3/_2$
x	1	0	$^1/_4$	$^1/_4$	0	4
s_3	0	0	$^3/_{16}$	$-^1/_{16}$	1	$-^1/_2$
$-z$	0	0	$-^{31}/_{16}$	$-^{11}/_{16}$	0	$-^{47}/_2$

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Step 4: Solve using the dual simplex method

	x	y	s_1	s_2	s_3	
y	0	1	0	0	-1	2
x	1	0	1	0	4	2
s_2	0	0	-3	1	-16	8
$-z$	0	0	-4	0	-11	-18

Solving the first two subproblems

What we get when we branch on $y \leq 1$ versus $y \geq 2$:

$$(x, y) = (4, 1.5)$$
$$z = 23.5$$

$y \geq 2$:

$$(x, y) = (2, 2)$$
$$z = 18$$

$y \leq 1$:

$$(x, y) = (3.\overline{3}, 1)$$
$$z = 18.\overline{3}$$

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- The left node is an integer solution, giving us a **lower bound** of 18.

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- The right node is a fractional solution with $z > 18$, so it's still worth exploring.

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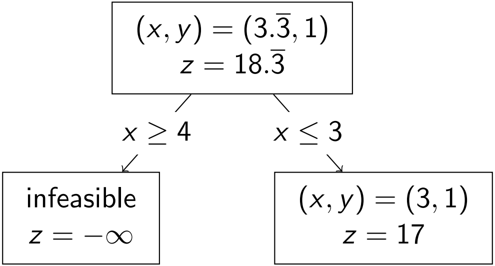
- The left node is an integer solution, giving us a **lower bound** of 18.
- The right node is a fractional solution with $z > 18$, so it's still worth exploring.
- We can branch on x: add $x \leq 3$ or $x \geq 4$ as a constraint.

Solving the next two subproblems

What we get when we branch on $x \leq 3$ versus $x \geq 4$ (from the node where we already had $y \leq 1$ as an extra constraint):



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- The left node is infeasible, so we ignore it completely.

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$x \geq 4$ → infeasible, $z = -\infty$

$x \leq 3$ → $(x, y) = (3, 1)$, $z = 17$

- The left node is infeasible, so we ignore it completely.
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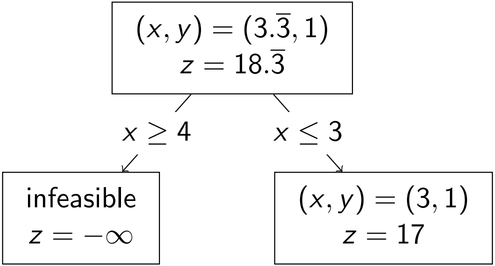

- The left node is infeasible, so we ignore it completely.
- The right node is another integer solution, but it has $z = 17 < 18$, so it's not as good as the first. (Even if it were a fractional solution, we wouldn't branch on it.)
- We have no more nodes worth exploring, so we're done.

A formal description

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We maintain:

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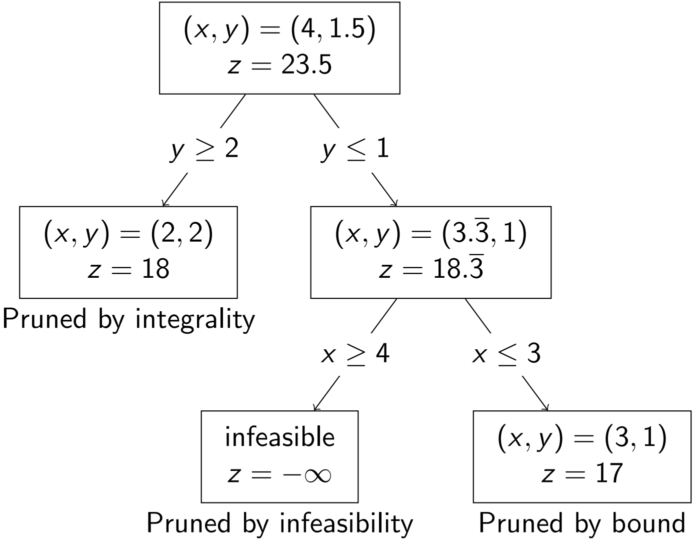
If the node we look at has no feasible solution, we also do nothing; the node is **pruned by infeasibility**.

Branching in our example



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- **Which fractional variable do we branch on, if we have a choice?**

  We might care if x_i is very close to an integer or far from one. We might also care if x_i has a high coefficient in the objective function.