Primal-Dual Algorithm II

Math 482, Lecture 30

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The four linear programs

The four linear programs in the primal-dual method:

$$(\mathbf{P}) \begin{cases} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{cases}$$

$$(\mathbf{RP}) \begin{cases} \underset{\mathbf{x}\in\mathbb{R}^n,\mathbf{y}\in\mathbb{R}^m}{\text{minimize}} & y_1 + \cdots + y_m \\ \text{subject to} & A_J\mathbf{x}_J + I\mathbf{y} = \mathbf{b} \\ & \mathbf{x}, \mathbf{y} \geq \mathbf{0} \end{cases}$$

$$(\mathbf{D}) \begin{cases} \underset{\mathbf{u}\in\mathbb{R}^m}{\text{maximize}} & \mathbf{u}^\mathsf{T}\mathbf{b} \\ \text{subject to} & \mathbf{u}^\mathsf{T}A \leq \mathbf{c}^\mathsf{T} \end{cases}$$

$$(\mathbf{DRP}) \begin{cases} \underset{\mathbf{v}\in\mathbb{R}^m}{\text{maximize}} & \mathbf{v}^\mathsf{T}\mathbf{b} \\ \text{subject to} & \mathbf{v}^\mathsf{T}A_J \leq \mathbf{0}^\mathsf{T} \\ & v_1, \ldots, v_m \leq 1 \end{cases}$$

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Here's what we have to figure out:

1. When we have the optimal solution to (**RP**), how do we find the optimal solution $\mathbf{v}$ to (**DRP**) (the augmenting direction)?
2. What is the benefit from considering (**RP**) instead of (**DRP**)?

Residual costs and dual solutions

Recall the formula:

$$r_i = c_i - \mathbf{c}_{\mathcal{B}}^{\mathsf{T}} A_{\mathcal{B}}^{-1} A_i = c_i - \mathbf{u}^{\mathsf{T}} A_i.$$

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Reduced cost of y_i is the slack in "$v_i \leq 1$" which is $1 - v_i$.

The old version and the new version

Previously, an iteration looked like:

1. Given a feasible solution $\mathbf{u}$ to $(\mathbf{D})$, check tightness of constraints to write down $(\mathbf{DRP})$.
2. Solve $(\mathbf{DRP})$ (in some way) and find an optimal direction $\mathbf{v}$.
3. Augment along $\mathbf{v}$ to get a better solution $\mathbf{u} + t\mathbf{v}$ to $(\mathbf{D})$; repeat.

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3. Augment along $\mathbf{v}$ just as before.

Example from previous lecture

$$(\mathbf{P})\begin{cases}\min & 2x_1 + 2x_2 + x_3 \\ \text{s. t.} & 2x_1 + x_2 - 4x_3 = 3 \\ & 4x_1 - x_2 + x_3 = 3 \\ & x_1, x_2, x_3 \geq 0\end{cases}$$

$$(\mathbf{D})\begin{cases}\max & 3u_1 + 3u_2 \\ \text{s. t.} & 2u_1 + 4u_2 \leq 2 \\ & u_1 - u_2 \leq 2 \\ & -4u_1 + u_2 \leq 1\end{cases}$$

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For the solution $\mathbf{u} = (1, 0)$ to $(\mathbf{D})$, we write down $(\mathbf{DRP})$, $(\mathbf{RP})$ as follows:

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Put (**RP**) into the tableau:

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Optimal direction: $\mathbf{v} = (1,1) - (0, \frac{3}{2}) = (1, -\frac{1}{2})$.

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To prove the lemma, we need to show: if an optimal solution to (**RP**) has $x_i > 0$, then x_i won't disappear from (**RP**) in the next iteration.

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To prove the lemma, we need to show: if an optimal solution to (**RP**) has $x_i > 0$, then x_i won't disappear from (**RP**) in the next iteration.

To use the lemma for good: use the previous optimal tableau to start solving (**RP**) in the next iteration.

Proof of lemma

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5. This constraint remains tight, so x_1 **remains in (RP)**.

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Written the new way:

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From one iteration of (RP) to the next

Suppose we solve this tableau to optimality:

	x_1	x_2	x_3	y_1	y_2	
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As before:

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- We augment $\mathbf{u}$ to $\mathbf{u} + t\mathbf{v}$ for the largest t that keeps this feasible.
- In this case, the constraint $u_1 - u_2 \leq 2$ means we stop at $t = \frac{2}{3}$, getting $\mathbf{u} + \frac{2}{3}\mathbf{v} = (\frac{5}{3}, -\frac{1}{3})$ as our next point.

From one iteration of (**RP**) to the next

At $\mathbf{u} = (\frac{5}{3}, -\frac{1}{3})$, $2u_1 + 4u_2 \leq 2$ is still tight, but so is $u_1 - u_2 \leq 2$.

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For the next iteration of (**RP**), we solve this tableau to optimality.

Ending the primal-dual algorithm

Our final tableau in the second iteration of (**RP**):

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- The reduced costs of y_1, y_2 are both 1. So $\mathbf{v} = (0, 0)$, and we won't augment any further: our $\mathbf{u} = (\frac{5}{3}, -\frac{1}{3})$ is optimal.
- In this tableau, we've found a solution $\mathbf{x}$ to (**P**) which has $x_j = 0$ whenever the j^{th} constraint of (**D**) is slack. (Here, the third constraint $-4u_1 + u_2 \leq 1$ is slack, and $x_3 = 0$.)

Ending the primal-dual algorithm

Our final tableau in the second iteration of (**RP**):

	x_1	x_2	x_3	y_1	y_2	
x_2	0	1	-3	$2/3$	$-1/3$	1
x_1	1	0	$-1/2$	$1/6$	$1/6$	1
$-z_{rp}$	0	0	0	1	1	0

This indicates that we've reached an optimal solution!

- The reduced costs of y_1, y_2 are both 1. So $\mathbf{v} = (0, 0)$, and we won't augment any further: our $\mathbf{u} = (\frac{5}{3}, -\frac{1}{3})$ is optimal.
- In this tableau, we've found a solution $\mathbf{x}$ to (**P**) which has $x_j = 0$ whenever the j^{th} constraint of (**D**) is slack. (Here, the third constraint $-4u_1 + u_2 \leq 1$ is slack, and $x_3 = 0$.) By complementary slackness, $\mathbf{x} = (1, 1, 0)$ is optimal for (**P**).