Primal-Dual Algorithm

Math 482, Lecture 29

Misha Lavrov

April 17, 2020

The problem

Our goal: to solve the primal-dual pair of linear programs below.

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Motivation: the Ford–Fulkerson method, where a single augmenting steps changes many variables at once.

The direction-finding problem

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- **Answer 2**. We should also make sure we don't accidentally leave the feasible region.

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(Fine print: this only works if the coefficients in the objective function $\mathbf{u}^\mathsf{T}\mathbf{b}$ are nonnegative, but we can make sure that this holds.)

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In our example, we get:

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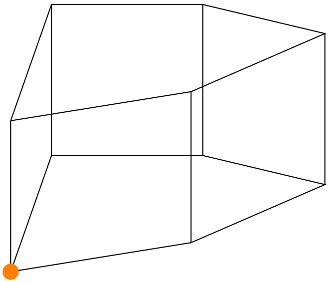
5. Replace $\mathbf{u}$ by $\mathbf{u} + t\mathbf{v}$ and go back to step 2. $\mathbf{u} + \frac{2}{3}\mathbf{v} = (\frac{5}{3}, -\frac{1}{3})$.

A made-up example

What this ideally looks like:



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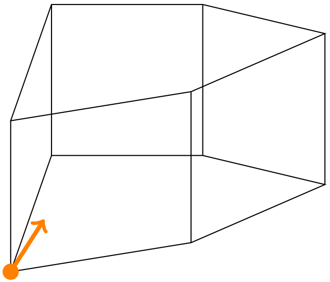

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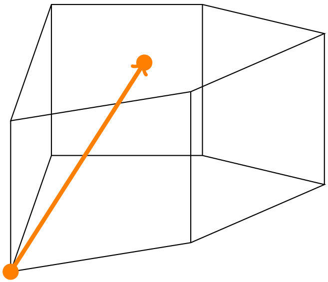

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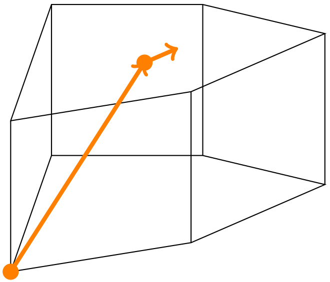

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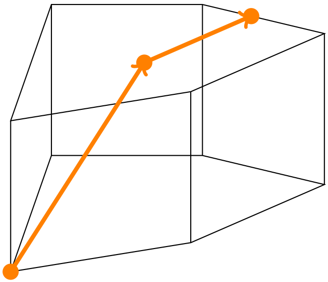

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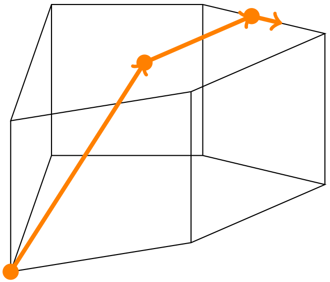

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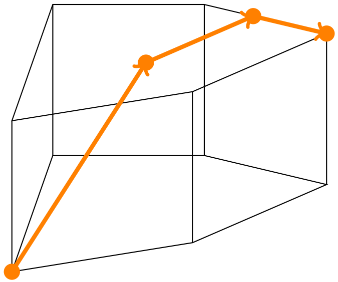

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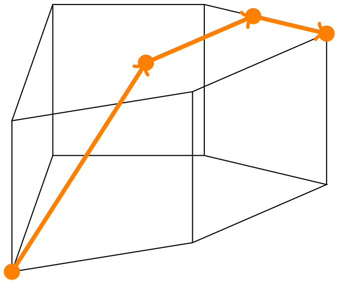

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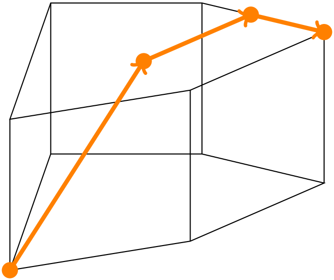

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Right now, the disadvantage is that each iteration requires solving its own LP. This is way too slow!

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Independent motivation: (**RP**) has an objective value of 0 if and only if $A_J \mathbf{x}_J = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ has a solution, which is the complementary slackness condition to see if $\mathbf{u}$ is optimal.

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- Solving (**DRP**) requires starting from scratch every time: whatever the optimal direction **v** was at the previous iteration, it's definitely not valid any more.
- However, (**RP**) keeps its constraints the same, possibly adding or removing variables, and it turns out that the optimal solution to (**RP**) will be a valid starting point for the next iteration of (**RP**).