The Ford–Fulkerson Algorithm

Math 482, Lecture 26

Misha Lavrov

April 6, 2020

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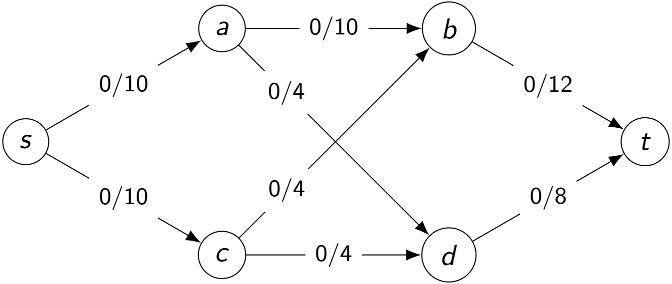

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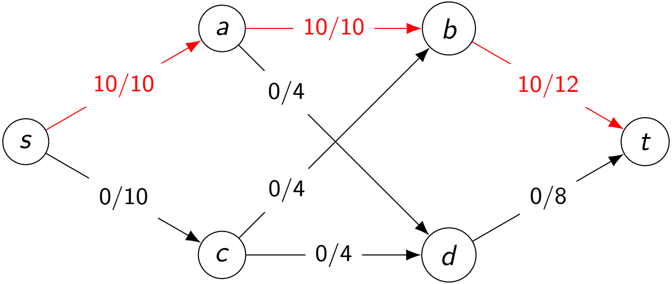

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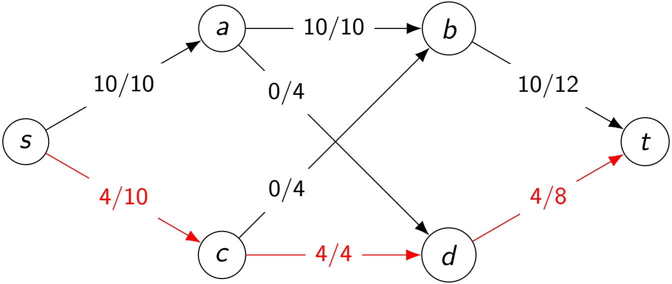

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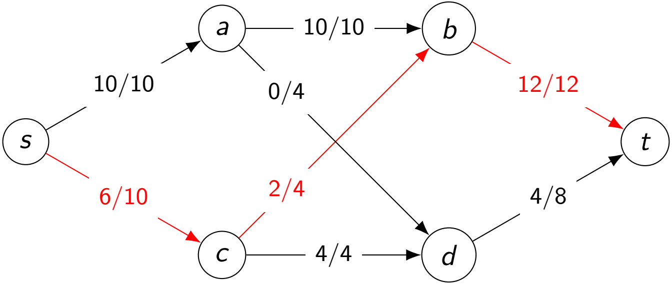

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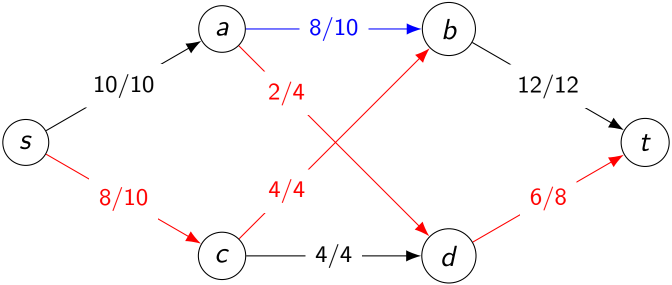

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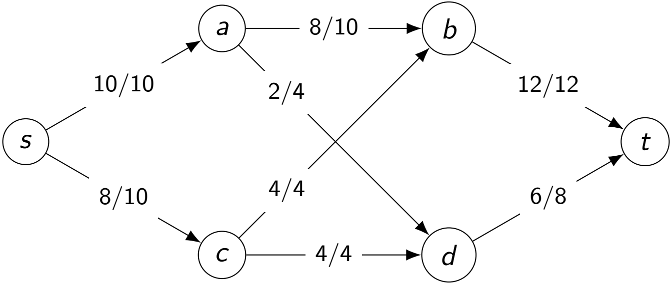
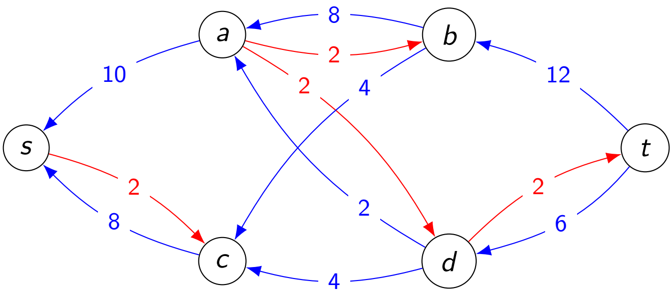


Eventually, there are no more augmenting paths.

The final residual graph

We can see this in the residual graph for the final flow obtained:



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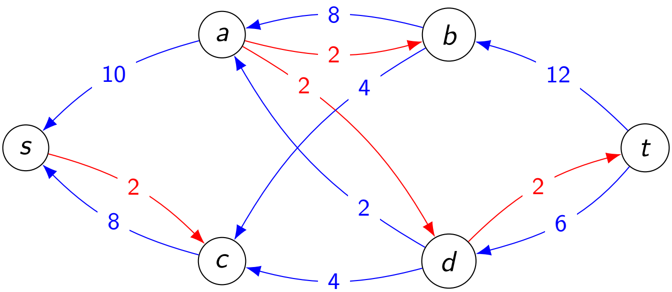

From s, we can only get to c. From c, we can't go anywhere new and can only return to s. **There is no s, t-path in the residual graph.**

The residual graph theorem

Theorem

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In our example, we take $S = \{s, c\}$ and $T = \{a, b, d, t\}$. The capacity of this cut is $c_{sa} + c_{cb} + c_{cd} = 10 + 4 + 4 = 18$, same as the value of $\mathbf{x}$.

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- Whenever $x_{ij} < c_{ij}$ for an arc $(i, j) \in A$, the residual graph has an arc $i \to j$.
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- For every arc (i, j) with $i \in T$ and $j \in S$, $x_{ij} = 0$.

Another equation for the value

Lemma

For **any** *cut* (S, T), $v(\mathbf{x}) = \sum_{i \in S} \sum_{j \in T} x_{ij} - \sum_{i \in T} \sum_{j \in S} x_{ij}$.

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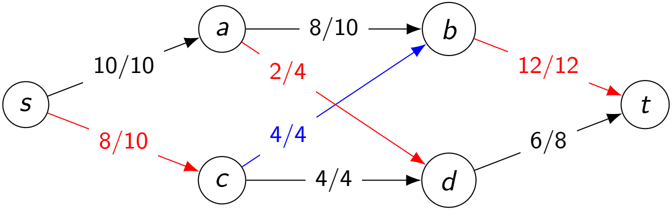

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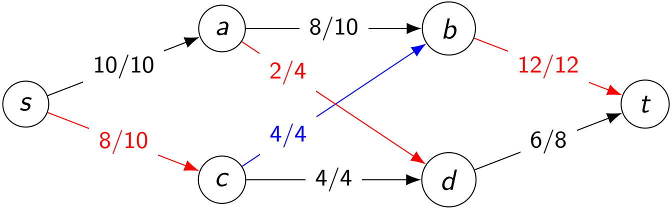

$$18 = v(\mathbf{x}) = x_{sc} + x_{ad} + x_{bt} - x_{cb} = 8 + 2 + 12 - 4.$$

Putting these together

If (S, T) is the cut from the residual graph, we still have

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$$v(\mathbf{x}) = \sum_{i \in S} \sum_{j \in T} c_{ij} - \sum_{i \in T} \sum_{j \in S} 0 = c(S, T).$$

This proves the residual graph theorem.

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One lingering doubt... how do we know that the algorithm will eventually stop?

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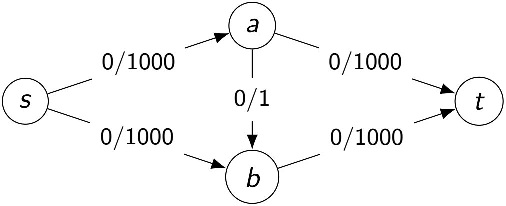

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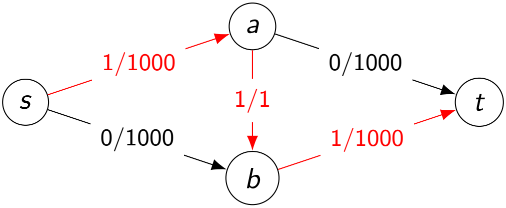

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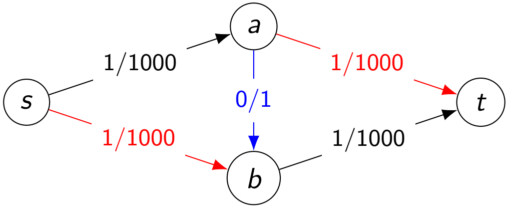

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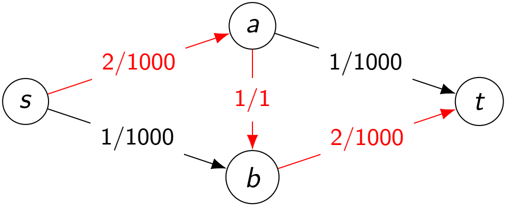

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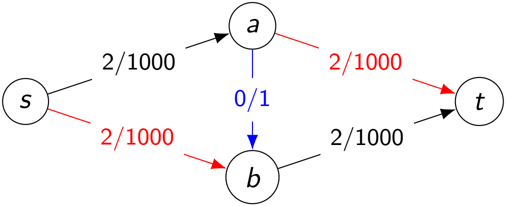
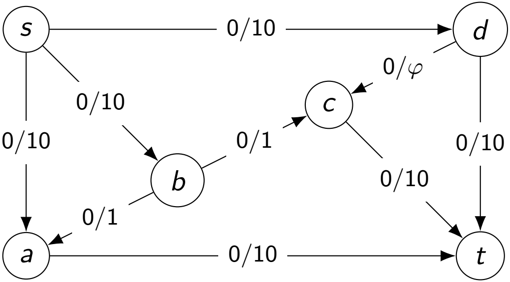


Infinite loop example

In general, if we pick our augmenting paths really badly, there are no guarantees. Example (see lecture notes for details):



One irrational capacity: $c_{dc} = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

The max value of 21 can be reached in 3 steps: augment along $s \to a \to t$, $s \to d \to t$, and $s \to b \to c \to t$. But it's possible to do infinitely many steps and be stuck at a value below 5.

Better guarantees and better algorithms

Suppose our network has n nodes and m arcs. (Note: $m < n^2$.)

- (Edmonds–Karp, 1972) Choose **the shortest augmenting path** at every step. Then at most nm augmenting steps are necessary: $O(nm^2)$ running time.

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- Modern state of the art: $O(nm)$ time, by choosing between two different algorithms when m is large or small.