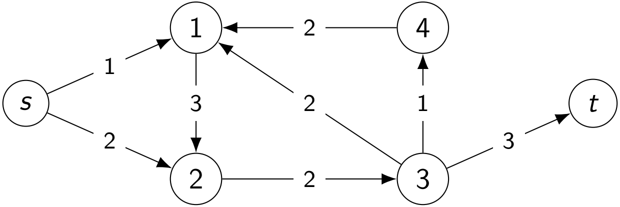
Network Flows

Math 482, Lecture 23

Misha Lavrov

March 30, 2020

Definition of a network



- A set N of *nodes*. Here, $N = \{s, 1, 2, 3, 4, t\}$.

  Node s is the *source* and node t is the *sink*.

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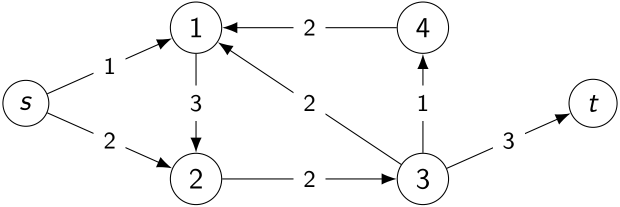

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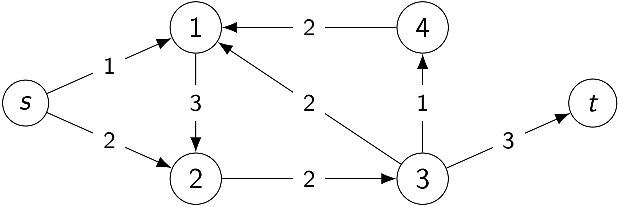

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  Here, $c_{s1} = 1$, $c_{12} = 3$, and so on.

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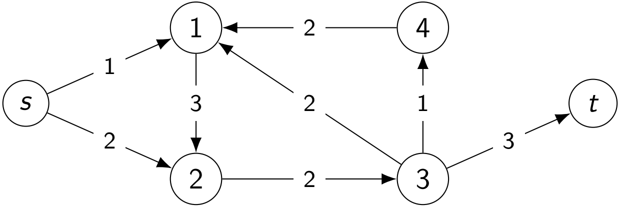

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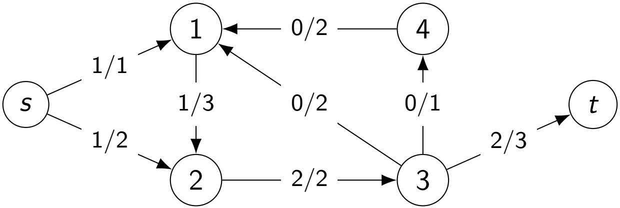
A flow **x** assigns a number x_{ij} to each arc $(i, j) \in A$.



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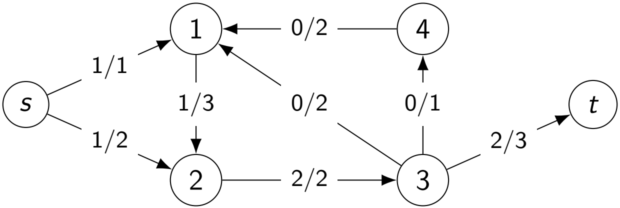

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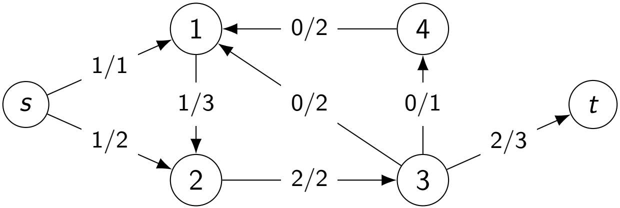

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- A flow represents stuff moving from s to t; x_{ij} is the amount of stuff moving along arc (i,j).
- For this to make sense, we want to add some constraints on $\mathbf{x}$ for it to be a feasible flow.

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Constraints on a feasible flow:

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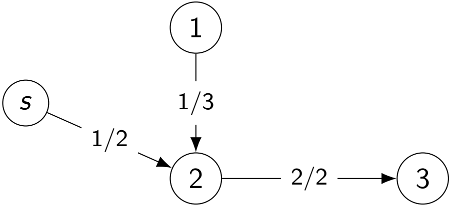
Constraints on a feasible flow:

- Capacity constraints: for every arc $(i, j) \in A$, $x_{ij} \le c_{ij}$.
- Nonnegativity constraints: $\mathbf{x} \ge \mathbf{0}$.
- Flow conservation: at every node $k \in N$ **except for** s **and** t, the total flow going in is equal to the total flow going out.



At node $k = 2$, we must have $x_{s2} + x_{12} = x_{23}$. Here, $1 + 1 = 2$.

More on flow conservation

The *excess* at a node k is the difference between the total flow into k and the total flow out of k:

$$\Delta_k(\mathbf{x}) := \sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj}.$$

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We can prove that $\Delta_s(\mathbf{x}) = -\Delta_t(\mathbf{x})$: the amount of gain at t is equal to the amount of loss at s. (This should follow from flow conservation.)

The maximum flow LP

The maximum flow problem to find the feasible flow in a network with the maximum value can be written as a linear program:

$$
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\underset{\mathbf{x}\in\mathbb{R}^{|A|}}{\text{maximize}} \quad & \sum_{i:(i,t)\in A} x_{it} - \sum_{j:(t,j)\in A} x_{tj} \\
\text{subject to} \quad & \sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} = 0 \quad (k \in N, k \neq s, t) \\
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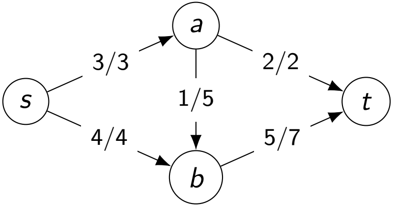
We can assume there are no arcs into s or out of t. In that case,

$$\text{value of } \mathbf{x} = \sum_{i:(i,t)\in A} x_{it} = \sum_{j:(s,j)\in A} x_{sj}.$$

How can we tell if a flow is optimal?

The flow in the example below has a value of 7. Can we do better?



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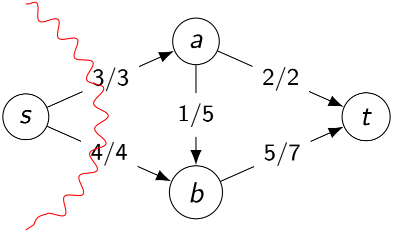

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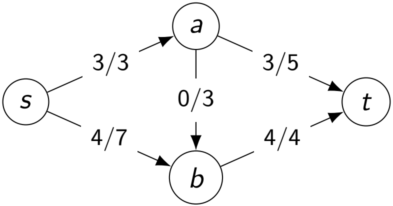

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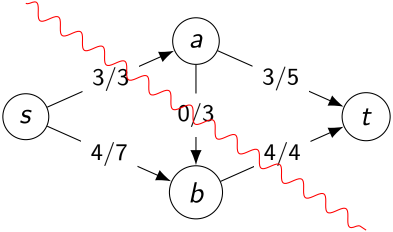

No: the arcs from $\{s, b\}$ to $\{a, t\}$ are all at their maximum capacity, and the arcs from $\{a, t\}$ to $\{s, b\}$ are all at capacity 0. We can't send more than 7 flow from $\{s, b\}$ to $\{a, t\}$.

Cuts

Definition

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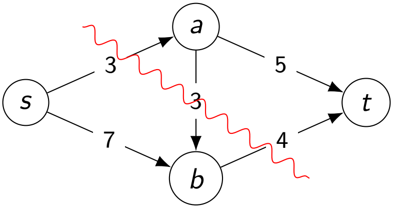

Here, $S = \{s, b\}$, $T = \{a, t\}$, and the capacity is $c_{sa} + c_{bt} = 7$.

Cuts are upper bounds on flows

Theorem

If a feasible flow $\mathbf{x}$ *has value* $v(\mathbf{x})$*, and a cut* (S, T) *has capacity* $c(S, T)$*, then*

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Proof idea: consider the sum

$$\sum_{k \in S} \left(\sum_{j:(k,j) \in A} x_{kj} - \sum_{i:(i,k) \in A} x_{ik} \right)$$

By computing this sum in two ways, we show that it is equal to $v(\mathbf{x})$, and also that it is at most $c(S, T)$.

Step 1

In the sum

$$\sum_{k \in S} \left(\sum_{j:(k,j) \in A} x_{kj} - \sum_{i:(i,k) \in A} x_{ik} \right)$$

the difference (in orange) is the net flow out of k. When $k \neq s$, it is 0. When $k = s$, it is the value of the flow.

Therefore

$$\sum_{k \in S} \left(\sum_{j:(k,j) \in A} x_{kj} - \sum_{i:(i,k) \in A} x_{ik} \right) = \sum_{j:(s,j) \in A} x_{sj} - \sum_{i:(i,s) \in A} x_{is} = v(\mathbf{x}).$$

Step 2

How many times, and with what sign, does x_{ij} appear in the sum

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We conclude that

$$v(\mathbf{x}) = \sum_{k\in S}\left(\sum_{j:(k,j)\in A} x_{kj} - \sum_{i:(i,k)\in A} x_{ik}\right) \leq c(S,T).$$