Favaron's Theorem, k-dependence, and Tuza's Conjecture

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Graph Theory Seminar
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Independence and Domination

Definition

$D \subseteq V(G)$ is independent if there are no edges between vertices of S.

$$\alpha(G) = \text{size of a largest independent set in } G.$$

$D \subseteq V(G)$ is dominating if every $v \in \overline{D}$ has a neighbor in D, where $\overline{D} = V(G) - D$.

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Observation (Ore 1962)

Every maximal independent set is a dominating set.
Thus, $\gamma(G) \leq \alpha(G)$ for any graph G.

k-Independence and k-Domination

Fink and Jacobson (1985) generalized these notions:

Definition

$D \subseteq V(G)$ is k-dependent if $\Delta(G[D]) \leq k-1$.

$$\alpha_k(G) = \text{size of a largest } k\text{-dependent set.}$$

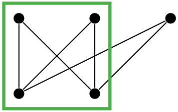
$D \subseteq V(G)$ is k-dominating if $|N(v) \cap D| \geq k$ for all $v \in \overline{D}$.

$$\gamma_k(G) = \text{size of a smallest } k\text{-dominating set.}$$

$$D \text{ is independent} \iff D \text{ is } 1\text{-dependent}$$
$$D \text{ is dominating} \iff D \text{ is } 1\text{-dominating}.$$

Is $\gamma_k(G) \leq \alpha_k(G)$?

If $k > 1$, a largest k-dependent set need not be k-dominating, e.g. $k = 3$:



Question (Fink–Jacobson)

Is $\gamma_k(G) \leq \alpha_k(G)$ for all G?

Fink and Jacobson proved the $k = 2$ case but left $k > 2$ open.

Favaron's Theorem

Definition

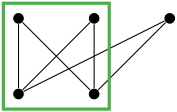
For $D \subseteq V(G)$, let $\phi_k(D) = k\,|D| - |E(G[D])|$.
A k-dependent set is k-optimal if it maximizes ϕ_k over all k-dependent sets.

Theorem (Favaron 1985)

If $D \subseteq V(G)$ is k-optimal, then D is k-dominating.

Corollary

Every graph G has a k-dominating k-dependent set.
In particular, $\gamma_k(G) \leq \alpha_k(G)$ for all G.



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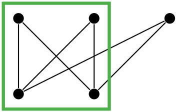

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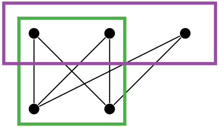

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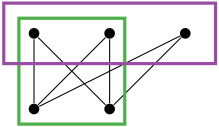

$\phi_3(D_2) = 3(3) - 0 = 9$

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Proof of Favaron's Theorem

Lemma

For any $D \subseteq V(G)$, there is a k-dependent $D' \subseteq D$ with $\phi_k(D') \geq \phi_k(D)$.

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Suppose that D is not k-dominating. Take $v \in \overline{D}$ with $|N(v) \cap D| < k$.

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Suppose that D is not k-dominating. Take $v \in \overline{D}$ with $|N(v) \cap D| < k$.

$$\phi_k(D \cup \{v\}) = \phi_k(D) + k - |N(v) \cap D| > \phi_k(D).$$

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$$\phi_k(D \cup \{v\}) = \phi_k(D) + k - |N(v) \cap D| > \phi_k(D).$$

Hence $D \cup \{v\}$ has a k-dependent subset D' with $\phi_k(D') > \phi_k(D)$, so D is not k-optimal. □

Matchings and Hall's Theorem

Definition

A *matching* M in a graph G is a set of pairwise disjoint edges.

$$V(M) = \{v \in V(G) \colon\ v \text{ lies in some edge of } M\}.$$

For disjoint $T, D \subseteq V(G)$, a *matching of T into D* is a matching M such that $X \subseteq V(M)$ and such that every edge of M has one endpoint in T and the other in D.

Theorem (Hall 1935)

An (T, D)-bigraph has a matching of T into D if and only if

$$|N(S)| \geq |S| \ \textit{ for all } S \subseteq T,$$

where $N(S) = \bigcup_{v \in S} N(v) \subseteq D$.

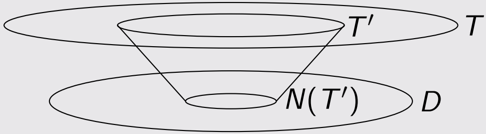
Maximum Independent Sets and Matchings

Theorem (Berge 1985)

An independent set D is maximum if and only if: for every independent T disjoint from D, there is a matching of T into D.

Proof of $\Rightarrow$.

If not, by Hall's Theorem there is $T' \subseteq T$ with $|N(T')| < |T|$.



Let $D' = D - N(T') \cup T'$. Now D' is independent with $|D'| > |D|$. $\square$

More Maximum Independent Sets and Matchings

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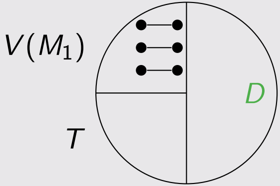

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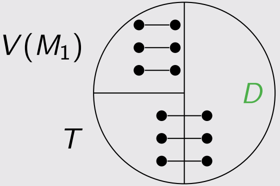

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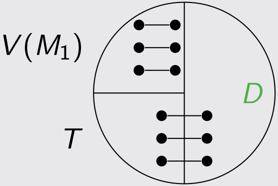

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T is independent, hence $\exists$ matching M_2 of T into D.
Now $\overline{D} \subseteq V(M_1 \cup M_2)$.

□

A k-Optimality Version of Berge's Theorem

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An independent set D is maximum if and only if: for every independent T disjoint from D, there is a matching of T into D.

Reformulation of $\Rightarrow$

If D is a 1-optimal set in a graph G, and T is an independent set disjoint from D, then there is a matching of T into D.

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Theorem (P. 2015+)

If D is a k-optimal set in a graph G, and T is an independent set disjoint from D, then there are k disjoint matchings of T into D.

Corollary

Every k-optimal set is k-dominating.

A Stronger k-Optimality Theorem

Definition

An *orientation* of a graph G is a digraph whose underlying graph is G.

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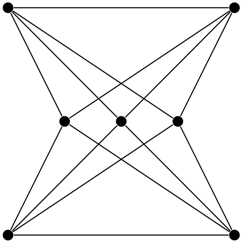
Main Theorem (P. 2015+)

Let D be a k-optimal set in a graph G, and let H be the subgraph of G with $E(H) = [D, \overline{D}]$.
If J is any orientation of $G[\overline{D}]$, then H has a k-edge-colorable subgraph M with

$$d^{+}(v) + d_{M}(v) \geq k \text{ for all } v \in \overline{D}.$$

Example ($k = 2$)



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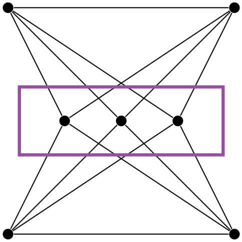

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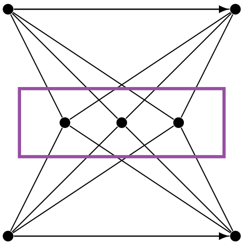

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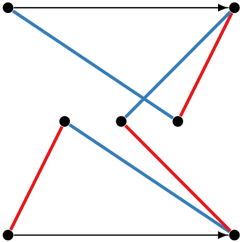

Proof of Main Theorem

Lemma (Multicolor Hall's Theorem)

Let G be an (X, Y)-bigraph and, for $x \in X$, let $d_x \geq 0$.
The following are equivalent:

1. *G has a k-edge-colorable subgraph M with $d_M(x) \geq d_x$ for all x,*
2. $\sum_{v \in Y} \min\{k, |N(v) \cap S|\} \geq \sum_{x \in S} d_x$ *for all $S \subseteq S$.*

When $k = 1$ and all $d_x = 1$, this is Hall's Theorem. When all $d_x = k$, this was proven by Lebensold (1977).

Proof of Main Theorem, Part 2

- Given *k*-optimal $D \subseteq V(G)$ and orientation of $G[\overline{D}]$.
- Want *k*-edge-cbl. $M \subseteq [D, \overline{D}]$ with $d_M(x) \geq k - d^+(x)$ for $x \in \overline{D}$.

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- If none such exists, then by Multicolor Hall's Theorem, get $S \subseteq \overline{D}$ such that $\sum_{v \in D} \min\{k, |N(v) \cap S|\} < \sum_{x \in S}(k - d^+(x))$.

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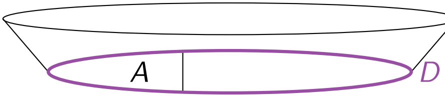

- Let $A = \{v \in D\colon\ |N(v) \cap S|\} \geq k$. Now

$$\sum_{v \in D} \min\{k, |N(v) \cap S|\} = k\,|A| + |[S, D - A]| < \sum_{x \in S}(k - d^+(x)).$$

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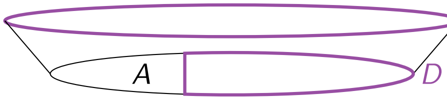

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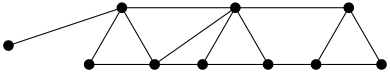
And Now for Something Completely Different

Question

Suppose I have a graph G, and I want to make G triangle-free by deleting a small number of edges. How many edges do I need to delete?



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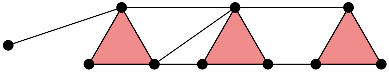

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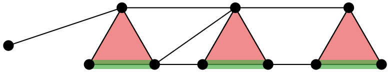

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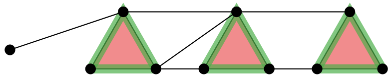

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To make G triangle-free, I must delete at least one edge from each triangle in $\mathcal{T}$.
If I delete all edges from the triangles in $\mathcal{T}$, then that will make G triangle-free, since $\mathcal{T}$ is maximal.

Tuza's Conjecture

Definition

$\tau(G) = \min\{|Y| : Y \subseteq E(G) \text{ and } G - Y \text{ is triangle-free}\}$.
$\nu(G) = \max\{|\mathcal{T}| : \mathcal{T} \text{ is a set of p.w. edge-disjoint triangles in } G\}$.

We have just shown that $\nu(G) \leq \tau(G) \leq 3\nu(G)$.

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Conjecture (Tuza 1981)

$\tau(G) \leq 2\nu(G)$ *for all graphs* G.

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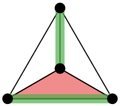
Conjecture (Tuza 1981)

$\tau(G) \leq 2\nu(G)$ *for all graphs* G.

Tuza's conjecture is sharp if true – consider K_4:



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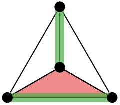


Theorem (Haxell 1999)

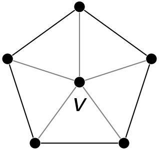
$\tau(G) \leq 2.87\nu(G)$ *for all graphs* G.

From Tuza's Conjecture to Matchings

Philosophy

Triangle packings are hard, but matchings are easy.

Suppose that $H = K_1 \vee G$, where G is triangle-free.



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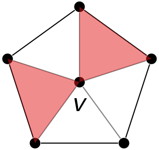

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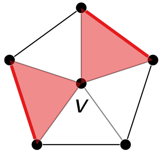

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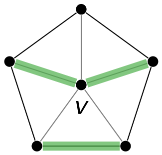

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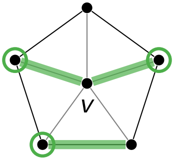

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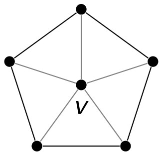

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Thus $\nu(H) = \alpha'(G)$ and $\tau(H) = \beta(G) = |V(G)| - \alpha(G)$.

From Tuza's Conjecture to Edge Coloring

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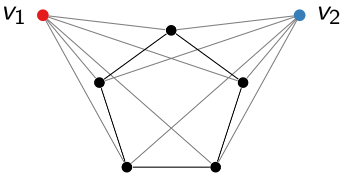
$\alpha'_k(G) = \max\{|E(H)| : H \text{ is a } k\text{-edge-colorable subgraph of } G\}.$



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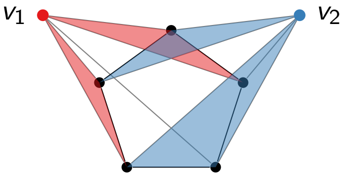

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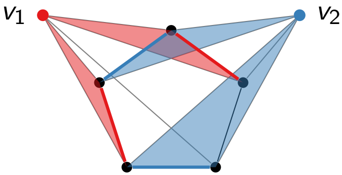

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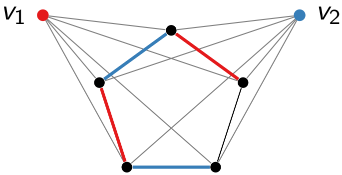

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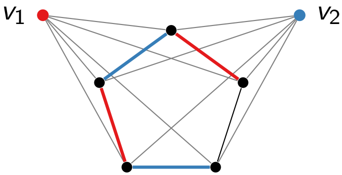

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$$\text{triangle packing in } H \leftrightsquigarrow k\text{-edge-colorable subgraph of } G.$$

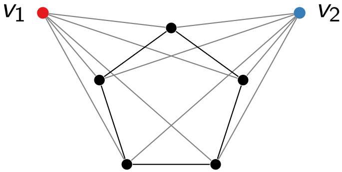
Thus, $\nu(H) = \alpha'_k(G)$.
Less obvious: we also have a correspondence

$$\text{triangle edge cover in } H \leftrightsquigarrow k\text{-dependent set in } G,$$

yielding $\tau(H) = k\,|V(G)| - \phi_k(G)$, where $\phi_k(G) = \max_{D \subseteq V(G)} \phi_k(D)$.

k-dependent Sets and Edge Cuts

Easy direction: $\tau(H) \leq k\,|V(G)| - \phi_k(G)$.



Given $D \subseteq V(G)$, let $X_D = \{vw\colon\ v \in I_k, w \notin D\} \cup E(G[D])$.

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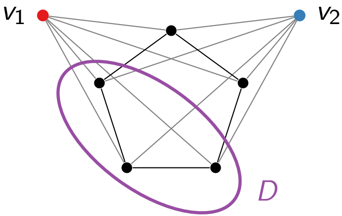

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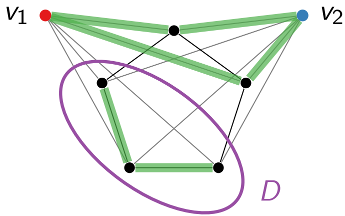

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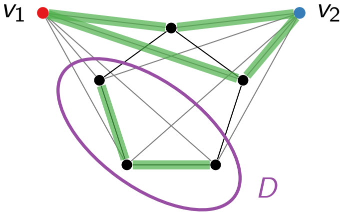

Given $D \subseteq V(G)$, let $X_D = \{vw\colon\ v \in I_k, w \notin D\} \cup E(G[D])$.
Now $H - X_D$ is triangle-free, and

$$|X_D| = k(|V(G) - D|) + |E(G[D])| = k\,|V(G)| - \phi_k(D).$$

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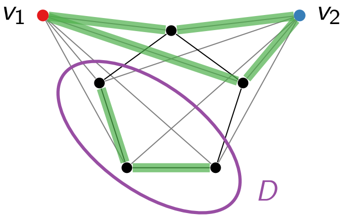

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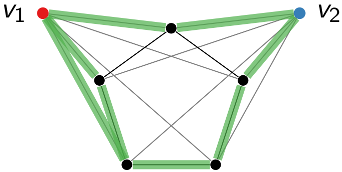
Taking a k-optimal D yields $\tau(H) \leq k\,|V(G)| - \phi_k(G)$.

k-dependent Sets and Edge Cuts

Slightly harder direction: $k\,|V(G)| - \phi_k(G) \leq \tau(H)$.

Idea: start with "optimal" triangle edge cover X, transform it into some X_D with $|X_D| \leq |X|$.



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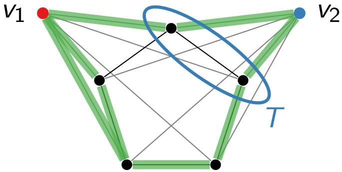

- For each i, let $T_i = \{v \in V(G)\colon\ v_i v \in X\}$. Let T be smallest T_i.

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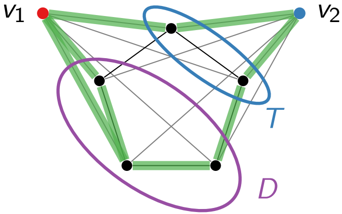

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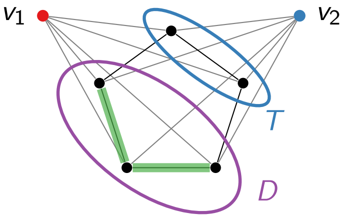

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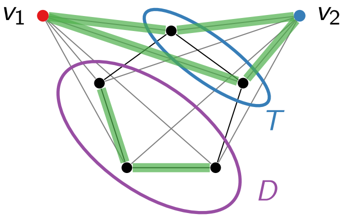

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- Since $\overline{T_i}$ is independent in $G - X_0$, have $X_D \cap E(G) \subseteq X_0$.
- Hence X_D is also a triangle edge cover, with

$$|X_D| \leq |X_0| + k\,|T| \leq |X_0| + \sum |T_i| = |X| = \tau(H).$$

A Special Case of Tuza's Conjecture

Conjecture (Tuza's Conjecture)

For any graph G, $\tau(G) \leq 2\nu(G)$.

Theorem

If G is triangle-free, then $\tau(I_k \vee G) = k\,|V(G)| - \phi_k(G)$ and $\nu(G) = \alpha'_k(G)$.

Thus, can reformulate Tuza's Conjecture for graphs of the form $I_k \vee G$:

Conjecture (Special Case of Tuza's Conjecture)

If G is triangle-free, then $2\alpha'_k(G) \geq k\,|V(G)| - \phi_k(G)$.

This conjecture seems natural even for non-triangle-free graphs.

A Conjecture

Corollary of Hall's Theorem

If D is a maximum independent set in a graph G, then G has a matching M with $\overline{D} \subseteq V(M)$.

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If D is a 1-optimal set, then G has a 1-edge-colorable subgraph M with $d_M(v) = 1$ for $v \in \overline{D}$.

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If D is a k-optimal set, then G has a k-edge-colorable subgraph M with $d_M(v) = k$ for $v \in \overline{D}$.

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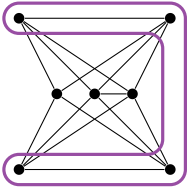
$$\Downarrow$$

Conjecture (Special Case of Tuza's Conjecture)

If G is triangle-free, then $2\alpha'_k(G) \geq k\,|V(G)| - \phi_k(G)$.

Degree-k Conjecture $\Rightarrow$ Special Case of Tuza's Conjecture

- Given k-optimal D, suppose G has k-edge-cbl. M with $d_M(v) = k$ for $v \in \overline{D}$. Let $M_0 = M \cap [D, \overline{D}]$.



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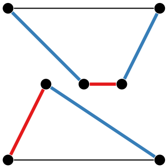

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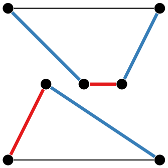

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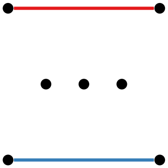

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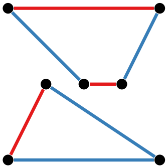

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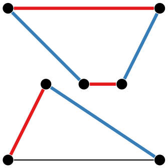

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- Uncolor each $G[D]$ edge conflicting with M_0. If q edges get uncolored, have $q \leq |M_0|$ and $q \leq |E(G[D])|$.
- How many edges still colored?

$$\alpha'_k(G) \geq |E(M)| + |E(G[D])| - q$$

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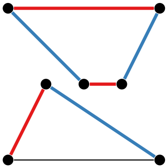

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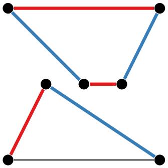

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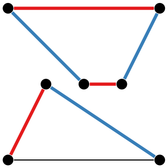
\alpha'_k(G) &\geq |E(M)| + |E(G[D])| - q \\
&\geq \frac{1}{2}\left(k\left|\overline{D}\right| + |E(G[D])|\right) \\
&= \frac{1}{2}\left(k\,|V(G)| - \phi_k(D)\right)
\end{aligned}$$

Degree-k Conjecture $\Rightarrow$ Special Case of Tuza's Conjecture



$k = 2$

- Given k-optimal D, suppose G has k-edge-cbl. M with $d_M(v) = k$ for $v \in \overline{D}$. Let $M_0 = M \cap [D, \overline{D}]$.
- Degree-sum formula: $k\left|\overline{D}\right| + |M_0| = 2\,|E(M)|$.
- By Vizing's Theorem, $G[D]$ is k-edge-cbl.
- Want to just combine colorings, but may have collisions.
- Uncolor each $G[D]$ edge conflicting with M_0. If q edges get uncolored, have $q \leq |M_0|$ and $q \leq |E(G[D])|$.
- How many edges still colored?

$$\begin{aligned}\alpha'_k(G) &\geq |E(M)| + |E(G[D])| - q \\ &\geq \frac{1}{2}\left(k\,|\overline{D}| + |E(G[D])|\right) \\ &= \frac{1}{2}\left(k\,|V(G)| - \phi_k(D)\right) \quad \checkmark\end{aligned}$$

A Related Result

Conjecture

For any graph G, we have $2\alpha'_k(G) \geq |V(G)| - \phi_k(G)$.

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Theorem

If G is chordal, then $2\alpha'_k(G) \geq |V(G)| - \phi_k(G)$.

Idea:

- Let D be a k-optimal set, and use a simplicial elimination order to obtain an orientation of $G[\overline{D}]$.
- Use main theorem and list-coloring tricks to obtain k-edge-chromatic subgraph M with $d_M(v) = k$ for $v \in \overline{D}$.
- Use Vizing's Theorem as before.

FIN