Beyond Ohba's Conjecture:

A bound on the choice number of k-chromatic graphs with n vertices

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slides available on DBW preprint page

Joint work with
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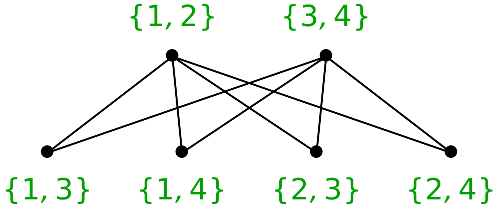
Ex. $\mathrm{ch}(K_{4,2}) > 2 = \chi(K_{4,2})$.



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Always $K_{2*(k-1),5*1}$ is not k-choosable (EOOS [2002])

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Also, $\left\lfloor \frac{8k}{5} \right\rfloor \le \mathrm{ch}(K_{5*k}) \le 2k$ and $\left\lfloor \frac{5k}{3} \right\rfloor \le \mathrm{ch}(K_{6*k}) \le \left\lceil \frac{7k-1}{3} \right\rceil$.

Lower Bound Constructions for $\mathrm{ch}(K_{m*k})$

Constr 1: Split $2k-1$ colors into $X_1, \ldots, X_m$. Assign all but X_i to the ith vertex in each part. L-coloring uses at least two colors on each part, in disjoint pairs. Hence it uses $2k$ colors, but only $2k-1$ exist. The list sizes are at least $\left\lfloor \frac{m-1}{m} 2k \right\rfloor$. ■

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Conj. (Noel [2012]) K_{m*k} has largest choice number among graphs with $\chi(G)=k$ and $n \le mk$.

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- Coloring procedure:
  (1) Break $V(G)$ into stable sets of size at most 2 by splitting some parts.
  (2) Produce an L-coloring whose color classes are these sets.

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Hall's Theorem picks distinct colors for the vertices outside X using colors outside $L(X)$. ■

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Minimality of G yields L'-coloring of G', which extends to L-coloring of G by using c on A. ■

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Lem. Each part A in G has size at most 4.

Pf. Since each color appears at most twice on A,

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Summary: (properties of minimal counterexample)

Obs. G is a complete multipartite graph.

Thm. (Noel–Reed–Wu) $n \ge 2k+2$, so $|L(v)| = \lceil\frac{n+k-1}{3}\rceil$.

Lem. (Kierstead) $|\bigcup_v L(v)| < n$.

Cor. Parts of size 2 have disjoint lists.

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(1) Obtain conditions that are sufficient for Hall's Theorem to guarantee the SDR.

(2) Define a procedure to make merges that guarantee these conditions.

A Sufficient Condition

Def. Let G have k_i parts of size i, for $i \in \{1, 2, 3, 4\}$.
Let A^* be what remains of part A after the merges.
Let Z_3 be a fixed set of $\left\lfloor \frac{2}{3} k_3 \right\rfloor$ 3-parts.
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(P4) $|A^*| = |A| = 3$ and $x, y \in A^*$ $\Rightarrow$ $|L(x) \cup L(y)| \geq k + k_3 + k_4$.
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(P8) The set of lists of merged vertices has an SDR.

Sufficiency

Lem. The lists left after good merges have an SDR.

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Now S is restricted to merged vertices, and (P8) suffices. ■

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Lem. These merges guarantee (P1)-(P7) if also each 3-part and each 4-part outside $Z_3 \cup Z_4$ has one merge. ■

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Def. A pair $\{u, v\}$ is a good pair for part A if
$|A| = \{u, v, w\}$ and $|L(u) \cap L(v)| \geq \left\lceil \frac{k_1+k_4+1}{3} \right\rceil$, or
$|A| = \{u, v, x, y\}$ and $|L(u) \cap L(v)| \geq |L(x) \cap L(y)|$.

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