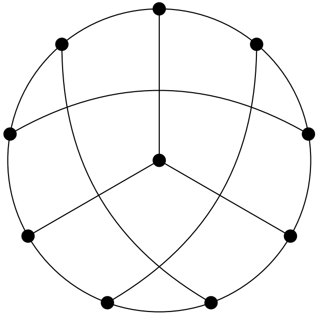
The minimum number of edges in a 4-critical graph that is bipartite plus 3 edges

A.V. Kostochka **B. Reiniger**

September 16, 2014
University of Illinois, Urbana-Champaign
Combinatorics Seminar

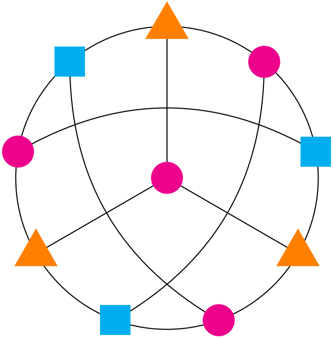
Graph coloring

A *proper k-coloring* of a graph G is an assignment $\phi : V \to [k]$ such that adjacent vertices receive distinct colors.
The minimum k such that G admits a proper k-coloring is the *chromatic number* of G, denoted $\chi(G)$.



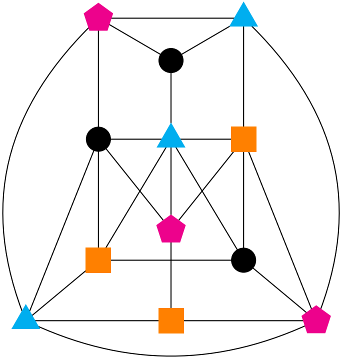
Graph coloring

A *proper k-coloring* of a graph G is an assignment $\phi : V \to [k]$ such that adjacent vertices receive distinct colors.
The minimum k such that G admits a proper k-coloring is the *chromatic number* of G, denoted $\chi(G)$.



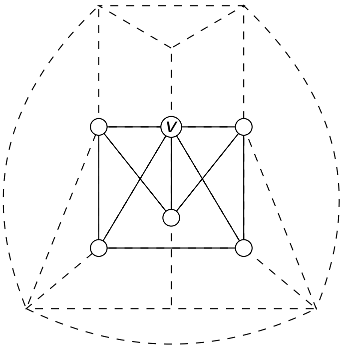
k-critical graphs

A graph G is called *k-critical* if $\chi(G) = k$ but every proper subgraph is properly $(k-1)$-colorable.



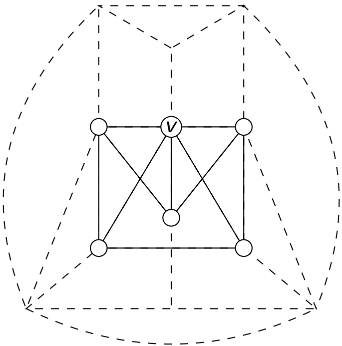
k-critical graphs

A graph G is called *k-critical* if $\chi(G) = k$ but every proper subgraph is properly $(k-1)$-colorable.



k-critical graphs

A graph G is called *k-critical* if $\chi(G) = k$ but every proper subgraph is properly $(k-1)$-colorable.



Every k-chromatic graph has a k-critical subgraph.

What can we say about k-critical graphs?

The only 1-critical graph is K_1.
The only 2-critical graph is K_2.
The only 3-critical graphs are odd cycles.
k-critical graphs should be neither too sparse nor too dense.

Question (Dirac, Erdős)

How many edges can an n-vertex k-critical graph have?

Theorem (Kostochka, Yancey)

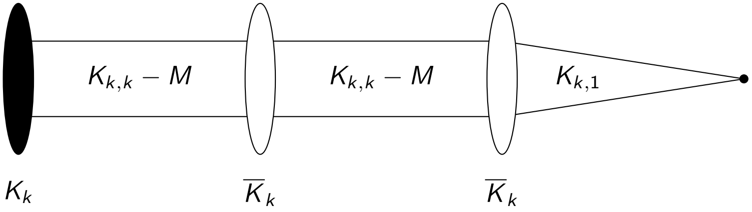
Every n-vertex k-critical graph has at least $\frac{(k+1)(k-2)n-k(k-3)}{2(k-1)}$ edges. This is sharp for $n \equiv 1 \pmod{k-1}$ when $k \geq 5$, as well as for all n when $k = 4$.

Nearly bipartite k-critical

Erdős conjectured that (for $k \geq 4$) k-critical graphs should not be "nearly bipartite," in the sense that for large n, deleting few edges from a k-critical graph should not make it bipartite.

Theorem (Rödl, Tuza)

For every $k \geq 2$ and infinitely many n, there is an n-vertex $(k+1)$-critical graph that can be made bipartite by deleting $\binom{k}{2}$ edges. This is best possible.



Main theorem

Say a graph is a $B + E_3$ *graph* if it can be obtained from a bipartite graph by adding three edges, and a $B + M_3$ *graph* if it can be obtained from a bipartite graph by adding a matching of three edges.

Chen, Erdős, Gyárfás, and Schelp found infinitely many 4-critical $B + M_3$ graphs, some with as few as $2n - 3$ edges total.

They conjectured that any 4-critical $B + M_3$ graph has many more than $5n/3$ edges, and they suggested that perhaps (asymptotically) $2n$ edges would be needed.

Main theorem

Say a graph is a $B + E_3$ *graph* if it can be obtained from a bipartite graph by adding three edges, and a $B + M_3$ *graph* if it can be obtained from a bipartite graph by adding a matching of three edges.

Chen, Erdős, Gyárfás, and Schelp found infinitely many 4-critical $B + M_3$ graphs, some with as few as $2n - 3$ edges total.

They conjectured that any 4-critical $B + M_3$ graph has many more than $5n/3$ edges, and they suggested that perhaps (asymptotically) $2n$ edges would be needed.

Theorem (Kostochka, R)

Any 4-critical n-vertex $B + E_3$ graph has at least $2n - 3$ edges.

List coloring

Define a restricted type of coloring as follows.

- Given a graph G, a *list assignment* L assigns to each $v \in V(G)$ a set of colors $L(v)$.
- G is *L-colorable* if there is a proper coloring ϕ such that for every v, $\phi(v) \in L(v)$.
- G is *k-choosable* if G is L-colorable for every list assignment L with $|L(v)| \geq k$ for all v.
- The *choosability* (a.k.a. *choice number* or *list chromatic number*) of G, denoted $\mathrm{ch}(G)$, is the minimum k such that G is k-choosable.

By assigning identical lists to all vertices, we see that $\mathrm{ch}(G) \geq \chi(G)$ for any G.

A curious connection

In a digraph, a *kernel* is an independent set S of vertices such that every vertex not in S has an edge into S. A digraph is *kernel-perfect* if every induced subdigraph has a kernel.

Theorem (Bondy, Boppana, and Siegel)

If D is a kernel-perfect digraph and L is a list assignment such that for every $v \in V(D)$, $|L(v)| \geq 1 + d^+(v)$, then D is L-colorable.

So we desire "regular" kernel-perfect orientations of graphs...

A curious connection

We will make use of the following old result of Hakimi. A weaker version of it is used by Alon and Tarsi.

Theorem (Hakimi)

Given a multigraph H and a function $f : V(H) \to \mathbb{N}$, one of the following holds.

1. *There is a subset $A \subseteq V(H)$ such that $|E(H[A])| > \sum_{v \in A} f(v)$.*
2. *There is an orientation of H such that for every $v \in V(H)$, $d^+(v) \leq f(v)$.*

(There is a simple proof by Hall's Theorem.)

Potential function

We define the *potential* of a set $A \subseteq V(G)$ as

$$\rho_G(A) = 2|A| - |E(G[A])|.$$

Our theorem is equivalent to $\rho_G(V(G)) \leq 3$ for all 4-critical $B + E_3$ graphs G.

Lemma (submodularity)

$\rho(A \cup B) + \rho(A \cap B) = \rho(A) + \rho(B) - |E(G[A - B, B - A])|.$

Proof idea

Let S denote the set of endpoints of the added 3 edges.
Let $P(G) = \min\{\rho_G(A) : A \supseteq S\}$.

If the theorem fails, consider a counterexample G with maximum $P(G)$, and subject to that, with fewest vertices.

Claim (Big Potentials)

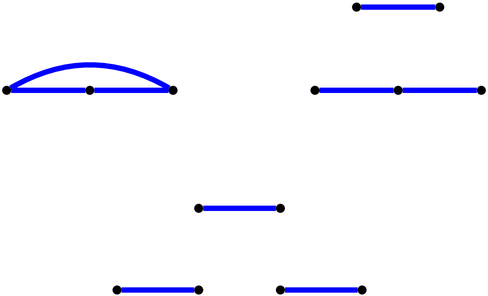
For this G, every subset of vertices containing S has large potential.

With $f(v) = 2$ for every $v \in V(G)$, Hakimi gives either

1. a set of vertices with small potential; we hope that this is impossible by the Big Potentials Claim; or
2. an orientation of G with all outdegrees at most 2; then we hope that Bondy, Boppana, and Siegel's result gives that G is 3-choosable.

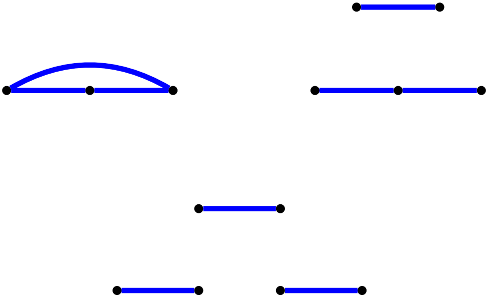
Some details

What can $G[S]$ be, if $G \neq K_4$?



Some details

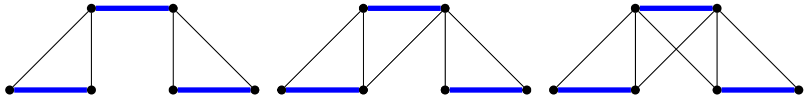
What can $G[S]$ be, if $G \neq K_4$?



There cannot be an independent set hitting every added edge (else G is 3-colorable).

Some details

What can $G[S]$ be, if $G \neq K_4$?

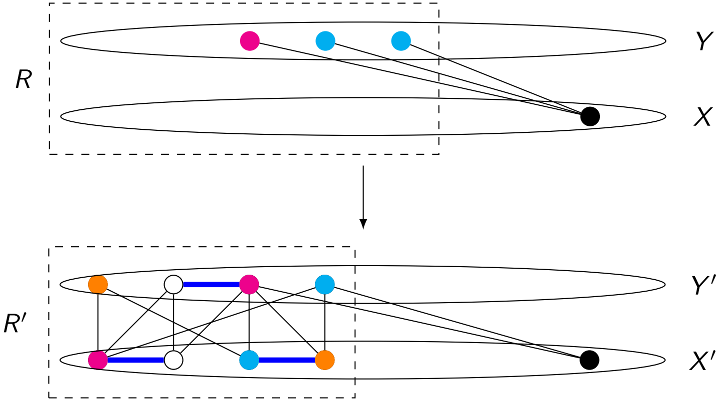


There cannot be an independent set hitting every added edge (else G is 3-colorable).

A replacement gadget

Claim (Big Potentials)

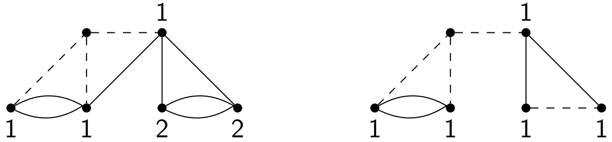
For our extremal choice of G, every set $R \supseteq S$ has potential at least 4. If furthermore $8 < |R| < |V(G)|$, then R has potential at least 5.



Getting kernel-perfect

Any orientation of a bipartite graph is kernel-perfect. Kostochka and Yancey proved that more generally, one can add antiparallel edges between vertices in one partite set and remain kernel-perfect.

To ensure the orientation coming from Hakimi is kernel-perfect, we first delete one vertex of S, double or remove some other edges, and define f appropriately.



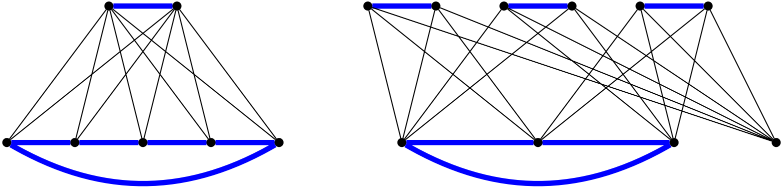
The Big Potentials Claim still gives us that the set in Hakimi's ❶ cannot exist, so we get a good orientation from Hakimi's ❷, which contradicts our choice of G.

Thank you!

Question

What about larger k? For instance, how few edges can a 5-critical $B + E_6$ graph have?

Rödl and Tuza's 5-critical constructions have $3n - 5$ edges, but there are some alternative constructions having only $3n - 6$.



Work partially supported by NSF grant DMS 08-38434, "EMSW21-MCTP: Research Experience for Graduate Students".