Online Ramsey games on planar graphs

Šárka Petříčková

Definition of the game

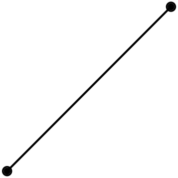
Originators: J. A. Grytczuk, M. Hałuszczak, H. A. Kierstead
(*On-line Ramsey Theory*, Electron. J. Combin. 11 (2004), #R75.)

- Let G be a simple undirected graph and $\mathcal{H}$ a family of graphs such that $G \in \mathcal{H}$.
- Online Ramsey game $(G, \mathcal{H})$ is a game between **Builder** a **Painter**, alternating in turns.
- In each turn Builder adds a new edge such that the built graph belongs to $\mathcal{H}$, and Painter colors it blue or red.
- **Builder wins** $(G, \mathcal{H})$ if she can force Painter to produce a monochromatic copy of G (for every Painter's play), and we say that G is **unavoidable** on $\mathcal{H}$.
- **Painter wins** if he can ensure that a monochromatic copy of G is never created. In this case we say that G is **avoidable** on $\mathcal{H}$.
- If every graph of $\mathcal{H}$ is unavoidable on $\mathcal{H}$, we say that $\mathcal{H}$ is **self-unavoidable**.

Note that without restriction to a certain class of graphs, Builder would always win by Ramsey's theorem.

Example

C_3 is unavoidable on planar graphs.



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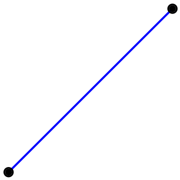

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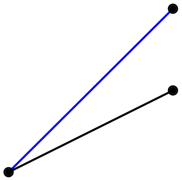

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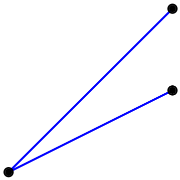

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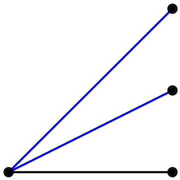

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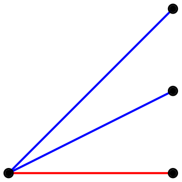

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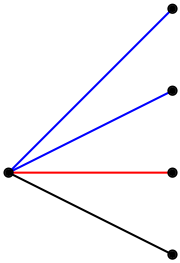

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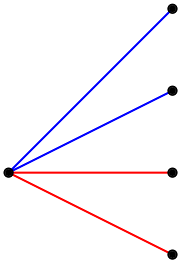

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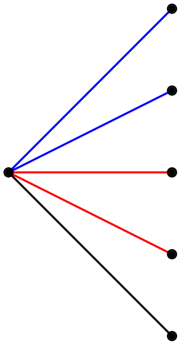

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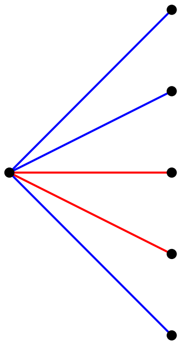

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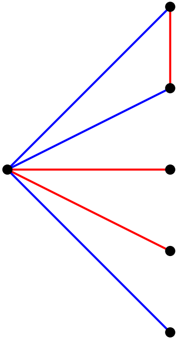

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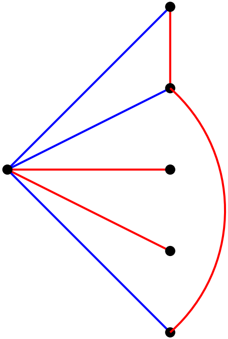

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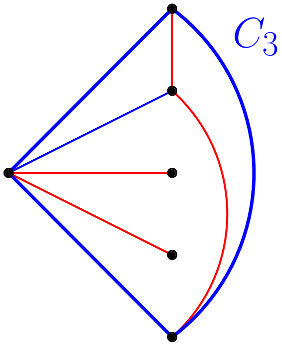

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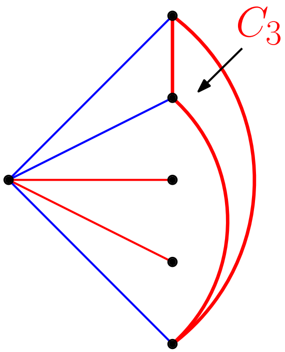


Example

C_3 is unavoidable on planar graphs.

How about C_n?

Known results

Grytczuk et al. [1]:

- The class of k-colorable graphs is self-unavoidable.
- The class of forests is self-unavoidable.
- C_3 is avoidable on the class of outerplanar graphs.
- C_3 in unavoidable on the class of planar 2-degenerate graphs.
- Every cycle is unavoidable on planar graphs.
- $K_4 - e$ is unavoidable on planar graphs.

Conjecture ([1])

G is unavoidable on planar graphs $\Leftrightarrow$ G is outerplanar.

Here we show:

- Theorem 1: G is unavoidable $\Leftarrow$ G is outerplanar.
- Theorem 2: G is unavoidable $\not\Rightarrow$ G is outerplanar.

Known results

Proposition ([1])

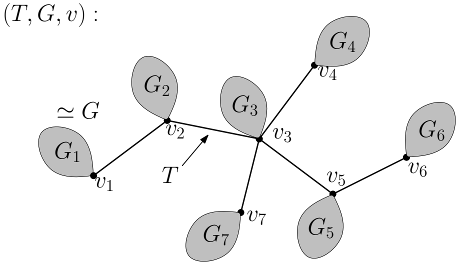
The class of forests is self-unavoidable.

Sketch of the proof:

- induction on the number of vertices n of the target tree T.
- for a leaf v of T we set $T' = T - v$
- T' is unavoidable on forests by the induction hypothesis, Builder forces n disjoint copies of T' of the same color
- then he builds a new copy of T by joining the vertices that correspond to the neighbour u of v in T

Easy observation

- Let T be a tree, G a graph, and $v \in V(G)$. We write (T, G, v) for a graph that arises after appending a copy of G to each vertex of T by v.



- unavoidable $:=$ unavoidable on the class of planar graphs

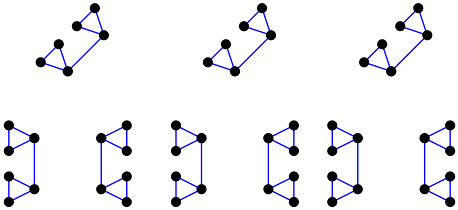
Easy observation

Lemma 1

If G is unavoidable, then (T, G, v) is unavoidable.

Proof:

- similar to the proof of the self-unavoidability of forests
- induction on the number of vertices of T
- f. e. $T = P_3$, $G = C_3$, $v \in V(G)$: Builder produces $|V((T, G, v))|$ copies of (P_2, C_3, v) of the same color



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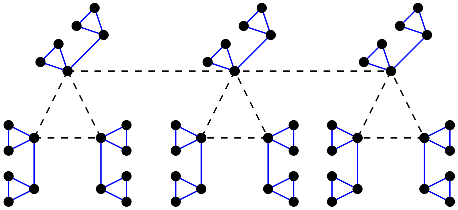

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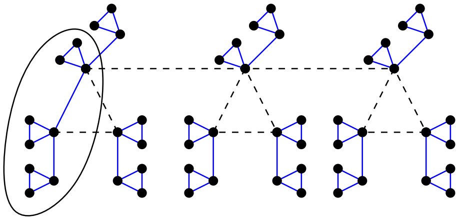
- f. e. $T = P_3$, $G = C_3$, $v \in V(G)$: if some of the new edges is blue:



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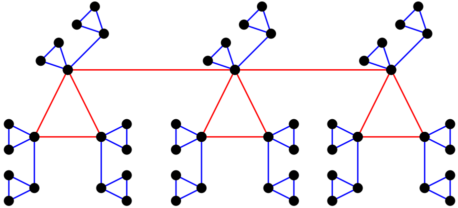
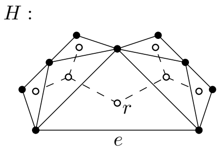
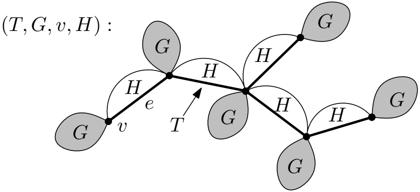


- Let H be an outerplanar graph such that there is a vertex $r \in V(H^*)$ such that H^* rooted in r is a full binary tree. We write (T, G, v, H) for a graph that arises from (T, G, v) if we append a copy of H to each edge of T by the "special" edge e.





Lemma 2

If G is outerplanar and unavoidable, then (T, G, v, H) is unavoidable.

Theorem 1

Every outerplanar graph is unavoidable.

Strategy $\mathbf{B}(\mathbf{T}, \mathbf{G}, \mathbf{v}, \mathbf{H}, \mathbf{X})$ for Builder:

Let $t = |V(T)|$ and $h = h(H)$.

1. If $t = 1$, call strategy $\mathbf{X}$ and stop.
2. If $h = 0$, call strategy $\mathbf{A}(T, G, v, \mathbf{X})$ and stop.
3. Choose a leaf u of T and call its neighbor u'. Call strategy $\mathbf{B}(T', G', v', H', \mathbf{X}')$, where
   - $T' = \overline{B}(T, G, v, H)$,
   - $G' = (T - u, G, v, H)$,
   - v' is the vertex of $T - u \subseteq G'$ that corresponds to u',
   - H' is the full outerplanar graph of height $h - 1$,
   - $\mathbf{X}' = \mathbf{B}(T - u, G, v, H, \mathbf{X})$.

   Let $\{u_1, \ldots, u_k\}$ be the vertex set of (T, G, v, H), and thus also a subset of a vertex set of $\overline{B}(T, G, v, H) = T'$. Adopt this notation to the subgraph T' of the winning copy (T', G', v', H') found by strategy $\mathbf{B}(T', G', v', H', \mathbf{X}')$. Add an edge $e_{ij} = u_i u_j$ in (T', G', v', H') if and only if $u_i u_j$ is an edge in (T, G, v, H).

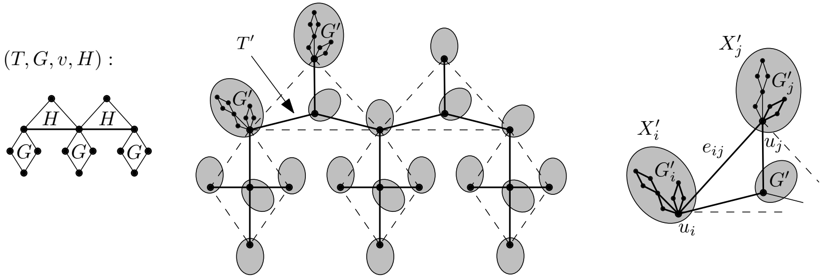


Figure : Forcing a monochromatic copy of (T, G, v, H), where T is a path of length 2, G is a cycle of length 4, v is any vertex of $V(G)$, and H is the full outerplanar graph of height 1.

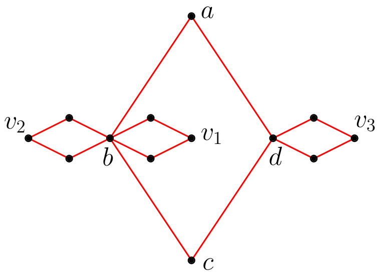
Unavoidable $\not\Rightarrow$ outerplanar

- $\theta_{i,j,k}$**-graph** is the union of three internally disjoint paths of lengths i, j, k that have the same two end vertices (for example, $K_{2,3}$ is $\theta_{2,2,2}$-graph).

Theorem 2

A $\theta_{2,j,k}$-graph is unavoidable for even j, k.

Proof for $j = k = 2$:



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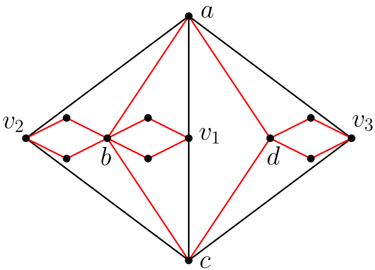

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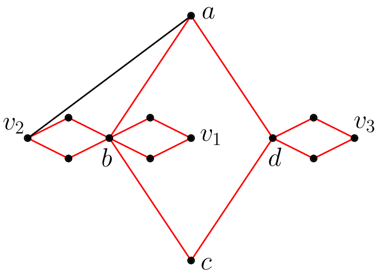

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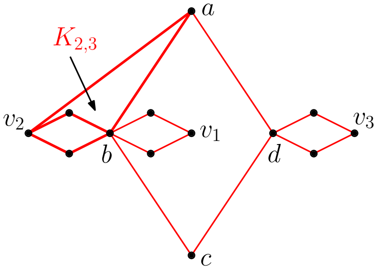

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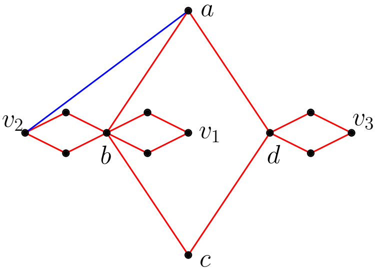

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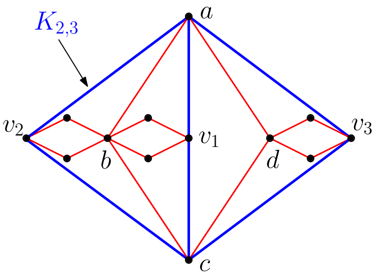

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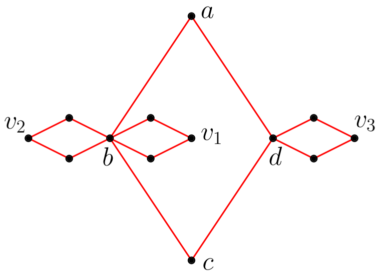

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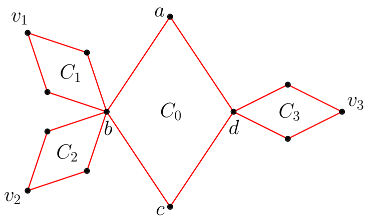
For $n = 0, \dots, 3$, let $G_n := C_0 \cup \cdots \cup C_n$.



Lemma 3

G_n is unavoidable ($n \in \{0, 1, 2, 3\}$).

Proof: induction on n:

- $n = 0$: special strategy
- $n > 0$: G_{n-1} unavoidable by induction hypothesis $\Rightarrow$ (T, G_{n-1}, v) unavoidable (by Lemma 1)

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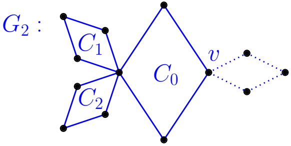

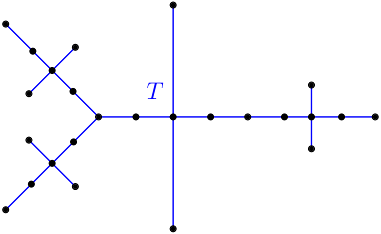

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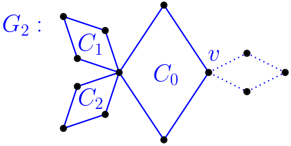

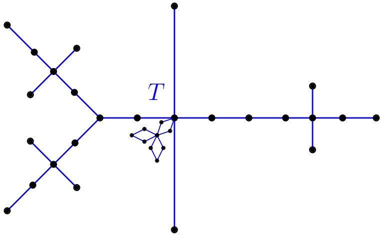

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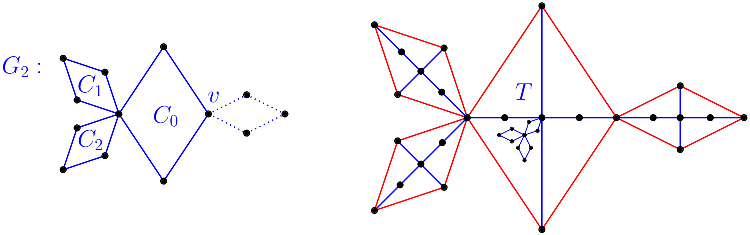

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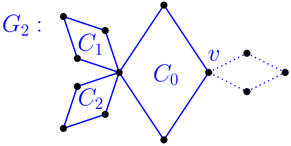
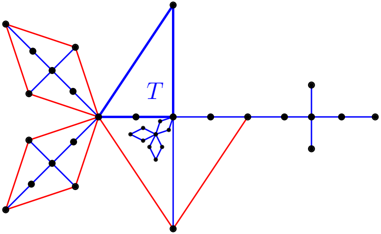


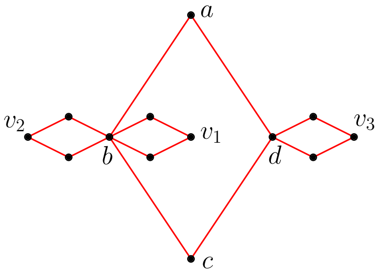


Unavoidable $\not\Rightarrow$ outerplanar

The strategy for forcing G_n ensures that every resulting graph can be embedded so that

(C1) all vertices $a, b, c, d, v_2, \ldots, v_n$ belong to some face f_1, and

(C2) a) either the vertices a, c, v_1 belong to some face f_2 ($f_2 \neq f_1$), or



b) there exists a vertex x not incident with f_1 such that there is a path $P = axc$ of the other color than G_n.

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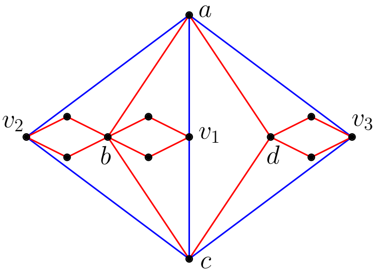

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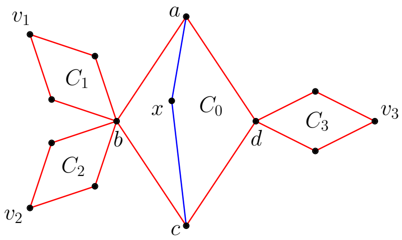

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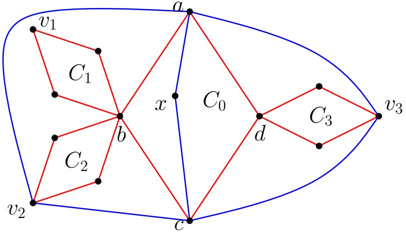

Open problems

- Is the class of planar graphs self-unavoidable?
- Does there exists an avoidable graph on planar graphs?
- In particular, is K_4 avoidable?
- What are another self-unavoidable classes?

Thank you.

Now it's time to do the taxes.

References

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