Characterization of $(2m, m)$-paintable graphs

Thomas Mahoney

University of Illinois at Urbana-Champaign
tmahone2@math.uiuc.edu

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Joint work with
Jixian Meng and Xuding Zhu

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Thm. (Meng–Puleo–Zhu 2014+) Characterized the 3-choice-critical graphs that are $(4,2)$-choosable.

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- Adjacent color sets are disjoint $\Rightarrow$ Proper (L, g)-coloring

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Thm. (Gutowski 2011) inf cannot be replaced by min!

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So v_{i-1}, v_i are marked and not colored $\geq\lceil m/k\rceil+1$ times.
Only happens once every $2k+1$ rounds.
$\Rightarrow$ All vxs lost $\lceil m/k\rceil(2k+1)+1 > t$ tokens. ■

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Main Thm. (MMZ 2014+) Given a graph G and $m \geq 1$, G is $(2m, m)$-paintable $\Leftrightarrow$ G is $(2, 1)$-paintable.

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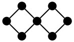
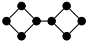
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .

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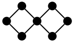
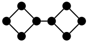
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .

- Using induction on the total number of tokens, give a winning strategy for Lister.

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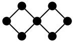
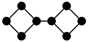
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .

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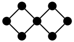
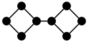
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .
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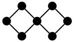
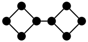
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .
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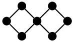
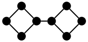
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .
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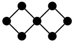
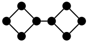
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .
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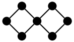
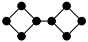
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .

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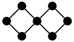
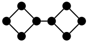
• Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .

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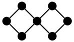
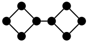
- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , and .

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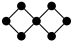
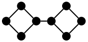
Appl. Any 3-paint-critical graph can be obtained by replacing edges with odd-length paths in one of $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , or .

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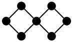
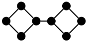
Appl. Any 3-paint-critical graph can be obtained by replacing edges with odd-length paths in one of $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}$, , or .

- Non-$(2m,m)$-paintability is preserved.

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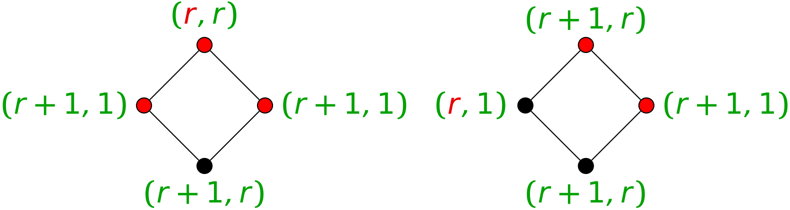
Special case: C_4

Lem. Lister can win in the following configurations:



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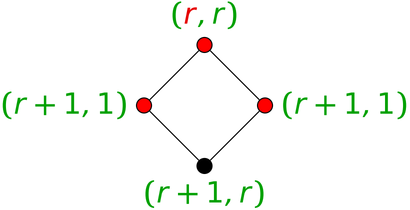
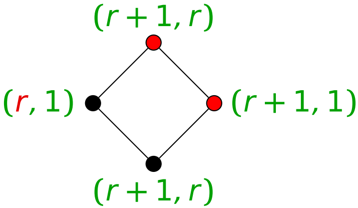
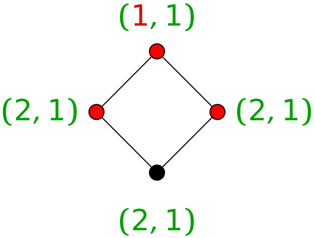




Pf. **Basis:** $r = 1$



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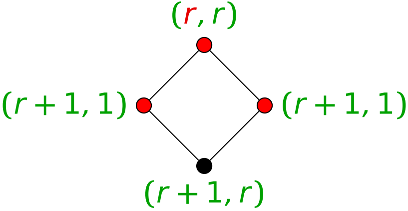

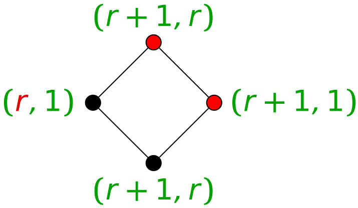

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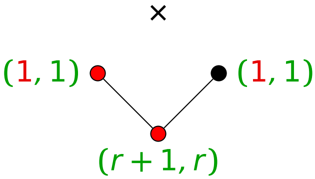

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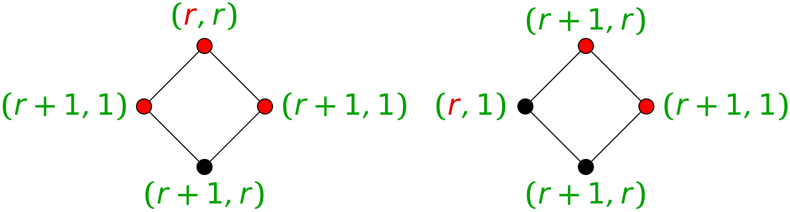

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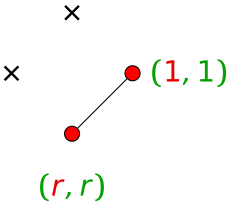

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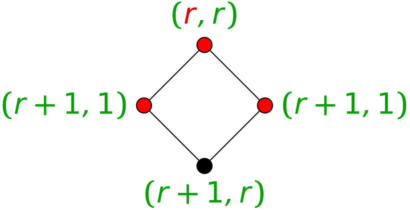

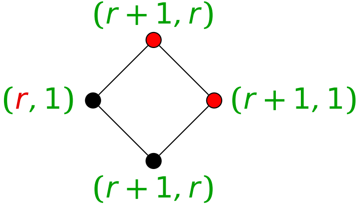

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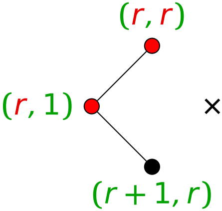

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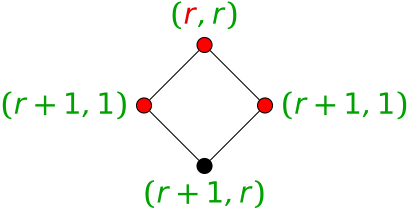

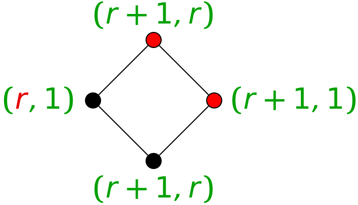

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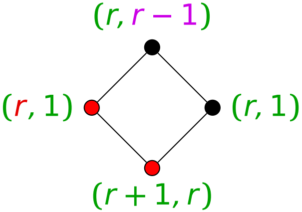

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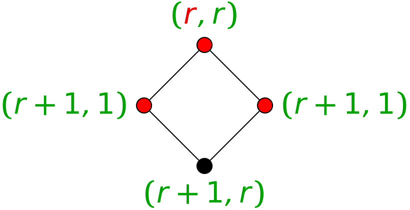

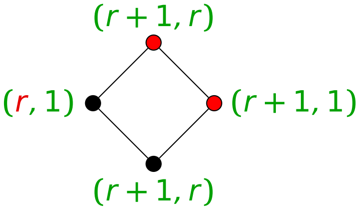

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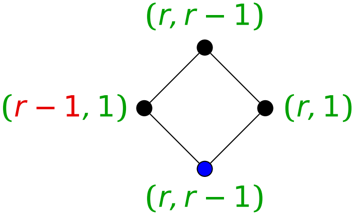

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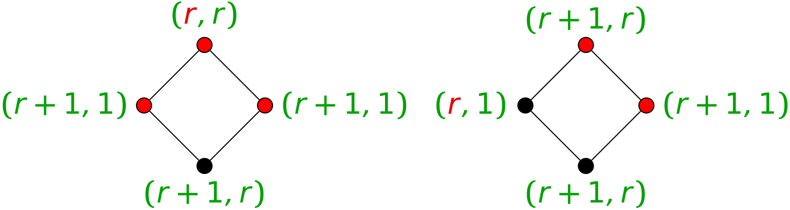

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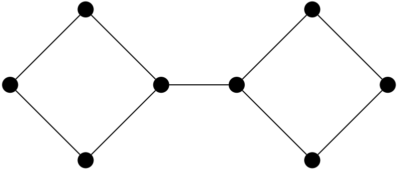

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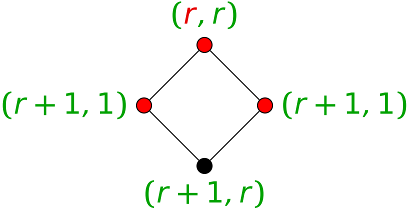

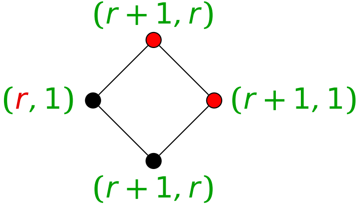

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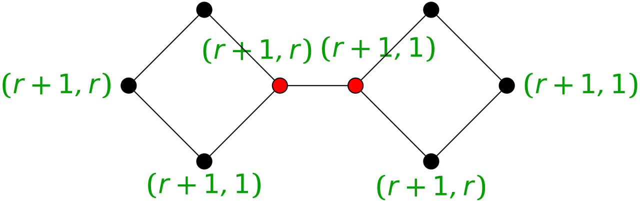

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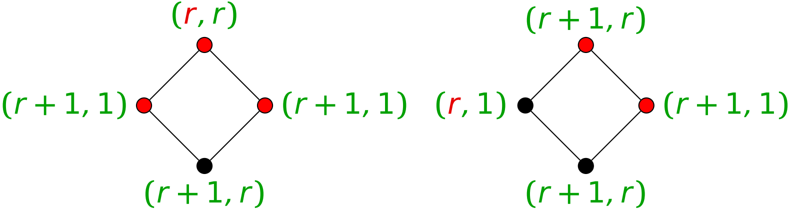
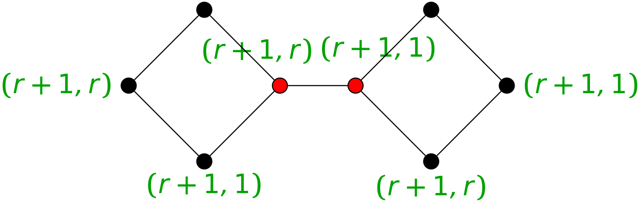
Special case: C_4

Lem. Lister can win in the following configurations:



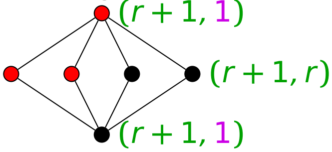
Cor. The following graph is not $(2m, m)$-paintable.



Pf. Lister marks all vertices until one partite set needs one more color. Then Lister marks the two 3-vertices, and wins on uncolored C_4 by the Lemma.

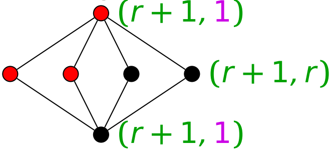
$K_{2,4}$ and Other Bipartite Cases

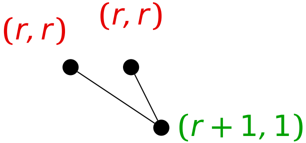
After Lister marks all vertices until one partite set is missing one color:



$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

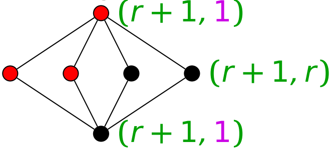


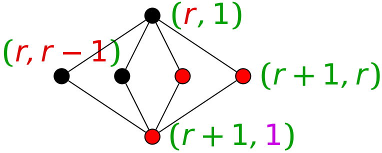


If Painter colors the top vertex, then Lister wins on

$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

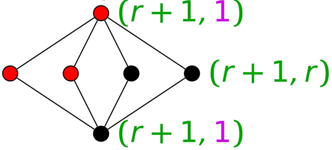


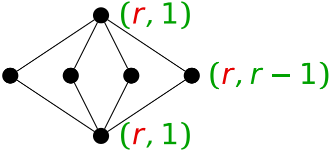


Otherwise, Lister marks the complement of the first set.

$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

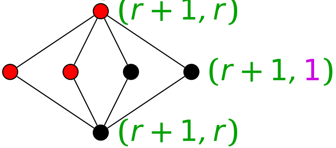




Using induction on r, we're done. (Basis: $\chi_p(K_{2,4}) > 2$)

$K_{2,4}$ and Other Bipartite Cases

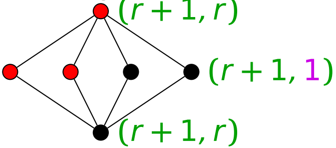
After Lister marks all vertices until one partite set is missing one color:



The other case for $K_{2,4}$ follows the same arguments. ∎

$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

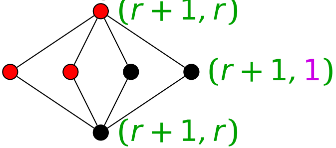


The other case for $K_{2,4}$ follows the same arguments. ■

For the remaining 3-paint-critical graphs:

$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:



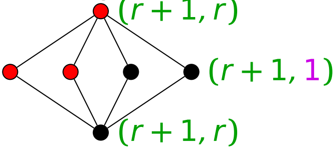
The other case for $K_{2,4}$ follows the same arguments. ∎

For the remaining 3-paint-critical graphs:

- Lister marks $V(G)$ until some part is missing one color.

$K_{2,4}$ and Other Bipartite Cases

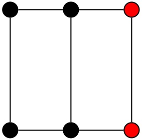
After Lister marks all vertices until one partite set is missing one color:



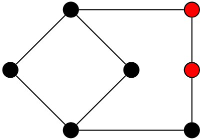
The other case for $K_{2,4}$ follows the same arguments. ■

For the remaining 3-paint-critical graphs:

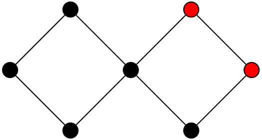
- Lister marks $V(G)$ until some part is missing one color.
- Lister plays to force a C_4 from the Lemma.



$\Theta_{1,3,3}$



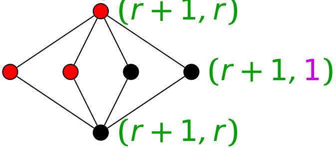
$\Theta_{2,2,4}$



$C_4 \cdot C_4$

$K_{2,4}$ and Other Bipartite Cases

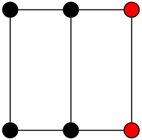
After Lister marks all vertices until one partite set is missing one color:



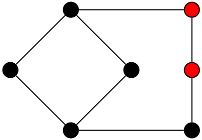
The other case for $K_{2,4}$ follows the same arguments. ■

For the remaining 3-paint-critical graphs:

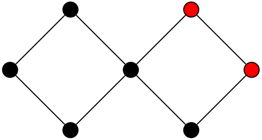
- Lister marks $V(G)$ until some part is missing one color.
- Lister plays to force a C_4 from the Lemma.



$\Theta_{1,3,3}$



$\Theta_{2,2,4}$



$C_4 \cdot C_4$

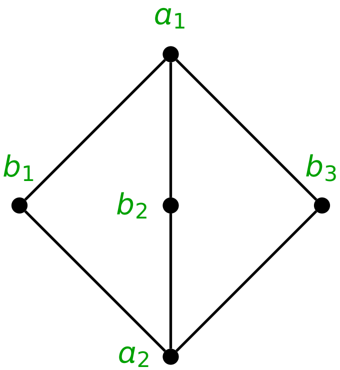
$\therefore$ $(2m, m)$-paintable $\Rightarrow$ $(2, 1)$-paintable.

$(2m, m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.

$(2m, m)$-Paintability of $K_{2,3}$

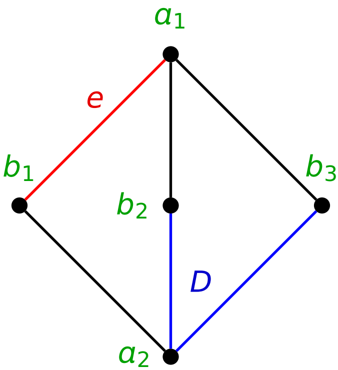
Given f, g, we define vertex names and weights.



$$w_A(a_i b_j) = f(a_i) - g(a_i b_j)$$
$$w_B(a_i b_j) = f(b_j) - g(a_i b_j)$$

$(2m, m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.



For an edge e and nonempty set D of edges disjoint from e, we require

$$(*) \quad \begin{aligned} w_A(e) + w_A(D) \geq 0 \\ w_B(e) + w_B(D) \geq 0. \end{aligned}$$

$$w_A(a_ib_j) = f(a_i) - g(a_ib_j)$$
$$w_B(a_ib_j) = f(b_j) - g(a_ib_j)$$

$(2m, m)$-Paintability of $K_{2,3}$

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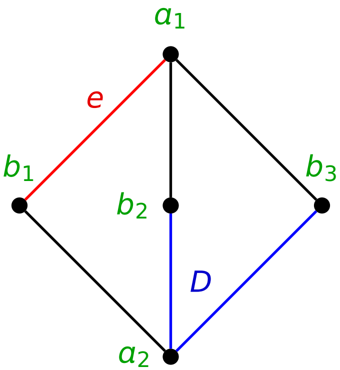

For an edge e and nonempty set D of edges disjoint from e, we require

(*) $$w_A(e) + w_A(D) \geq 0$$
$$w_B(e) + w_B(D) \geq 0.$$

A vertex v is forced if $f(v) = g(v)$.

$$w_A(a_ib_j) = f(a_i) - g(a_ib_j)$$
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$(2m, m)$-Paintability of $K_{2,3}$

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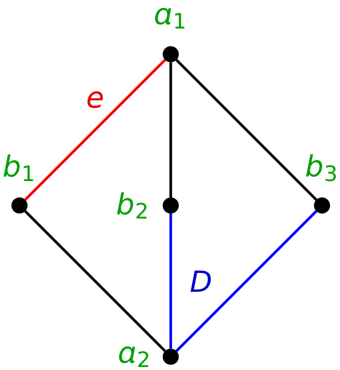

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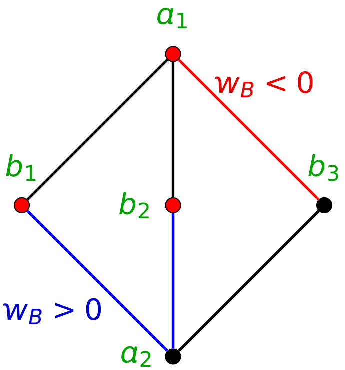
If Lister marks M and no vertex is forced, then Painter plays according to the following strategy:

$(2m, m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.



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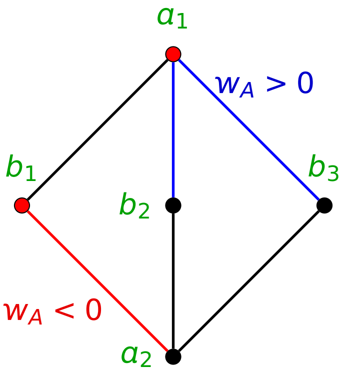
If Lister marks M and no vertex is forced, then Painter plays according to the following strategy:

1) $a_i \in M, b_j \notin M, w_B(a_ib_j) < 0$
$\Rightarrow$ Color $M \cap A$,

$(2m, m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.



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$\Rightarrow$ Color $M \cap A$,

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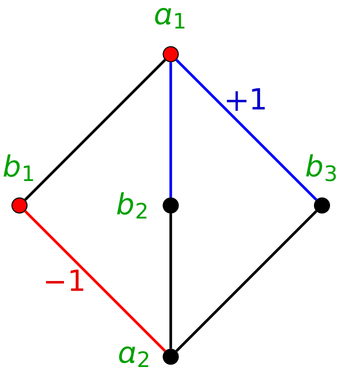
$\Rightarrow$ Color $M \cap B$,

$(2m, m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.



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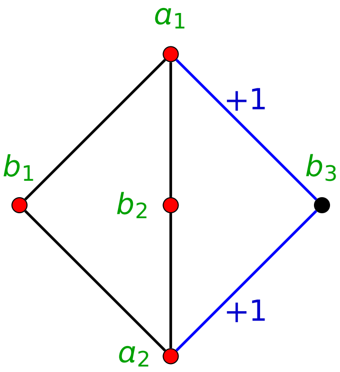
2) $a_i \notin M, b_j \in M, w_A(a_ib_j) < 0$ $\Rightarrow$ Color $M \cap B$,
3) $|M \cap A| \geq |M \cap B|$ $\Rightarrow$ Color $M \cap A$,

$(2m, m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.



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For an edge e and nonempty set D of edges disjoint from e, we require

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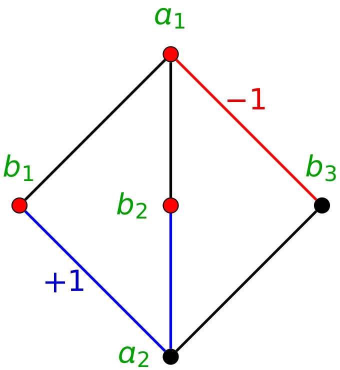
2) $a_i \notin M, b_j \in M, w_A(a_ib_j) < 0 \Rightarrow$ Color $M \cap B$,
3) $|M \cap A| \geq |M \cap B| \Rightarrow$ Color $M \cap A$,

$(2m, m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.



$w_A(a_i b_j) = f(a_i) - g(a_i b_j)$
$w_B(a_i b_j) = f(b_j) - g(a_i b_j)$

For an edge e and nonempty set D of edges disjoint from e, we require

$$(*) \quad w_A(e) + w_A(D) \geq 0$$
$$w_B(e) + w_B(D) \geq 0.$$

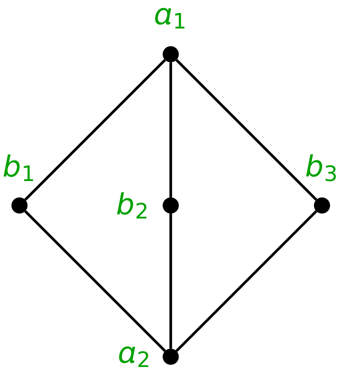
A vertex v is forced if $f(v) = g(v)$.

If Lister marks M and no vertex is forced, then Painter plays according to the following strategy:

1) $a_i \in M, b_j \notin M, w_B(a_i b_j) < 0$ $\Rightarrow$ Color $M \cap A$,
2) $a_i \notin M, b_j \in M, w_A(a_i b_j) < 0$ $\Rightarrow$ Color $M \cap B$,
3) $|M \cap A| \geq |M \cap B|$ $\Rightarrow$ Color $M \cap A$,
4) Otherwise, Color $M \cap B$.

$(2m,m)$-Paintability of $K_{2,3}$

Given f, g, we define vertex names and weights.



For an edge e and nonempty set D of edges disjoint from e, we require

$$(*) \quad \begin{array}{l} w_A(e) + w_A(D) \geq 0 \\ w_B(e) + w_B(D) \geq 0. \end{array}$$

A vertex v is forced if $f(v) = g(v)$.

If Lister marks M and no vertex is forced, then Painter plays according to the following strategy:

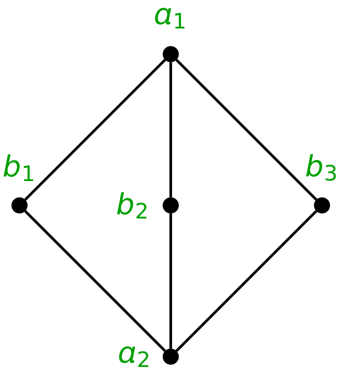
1) $a_i \in M, b_j \notin M, w_B(a_ib_j) < 0 \Rightarrow$ Color $M \cap A$,
2) $a_i \notin M, b_j \in M, w_A(a_ib_j) < 0 \Rightarrow$ Color $M \cap B$,
3) $|M \cap A| \geq |M \cap B| \Rightarrow$ Color $M \cap A$,
4) Otherwise, Color $M \cap B$.

$$w_A(a_ib_j) = f(a_i) - g(a_ib_j)$$
$$w_B(a_ib_j) = f(b_j) - g(a_ib_j)$$

Always, $(*)$ preserved and $\max\{w_A(e), w_B(e)\} \geq 0$.

$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

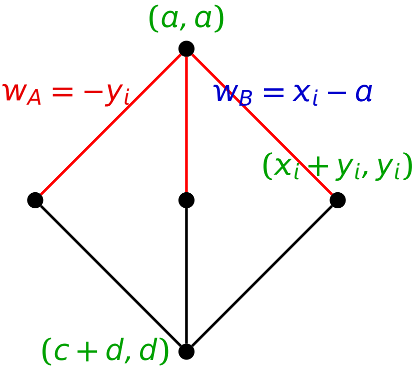
Painter uses strategy until a vertex is forced.



$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

Case 1: a_1 is forced.

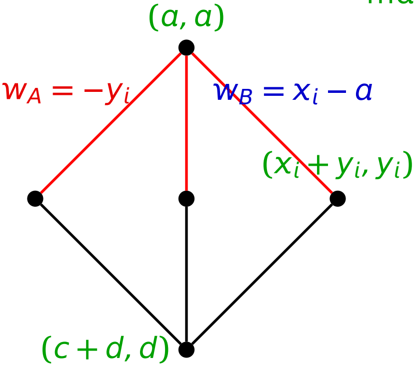


$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

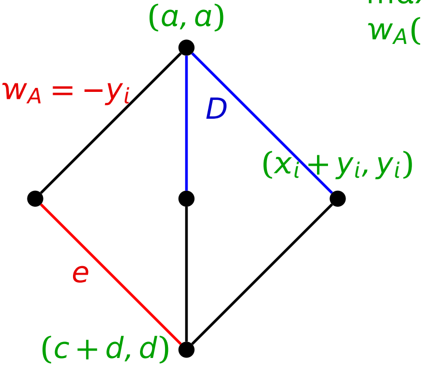
Case 1: a_1 is forced.

$\max\{w_A(e), w_B(e)\} \geq 0 \;\Rightarrow\; x_i \geq a$



$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.



Case 1: a_1 is forced.

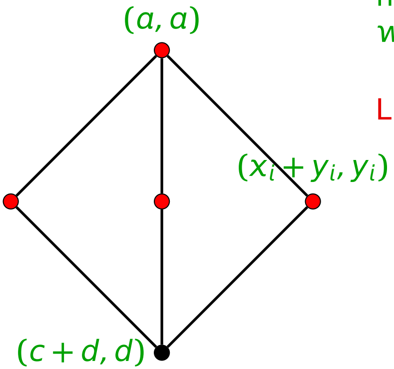
$\max\{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq a$

$w_A(e) = c - y_1, w_A(D) = -y_2 - y_3$

$\Rightarrow c \geq y_1 + y_2 + y_3$

$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.



Case 1: a_1 is forced.

$\max\{w_A(e), w_B(e)\} \geq 0 \;\Rightarrow\; x_i \geq a$

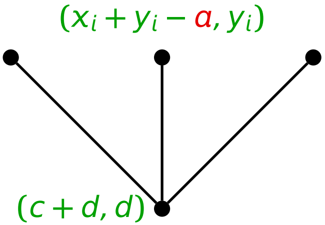
$w_A(e) = c - y_1, w_A(D) = -y_2 - y_3$

$\Rightarrow\; c \geq y_1 + y_2 + y_3$

Lister marks $N[a_1]$ for a rounds.

$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.



Case 1: a_1 is forced.

$\max\{w_A(e), w_B(e)\} \geq 0 \;\Rightarrow\; x_i \geq a$

$w_A(e) = c - y_1, w_A(D) = -y_2 - y_3$

$\Rightarrow\; c \geq y_1 + y_2 + y_3$

Lister marks $N[a_1]$ for a rounds.

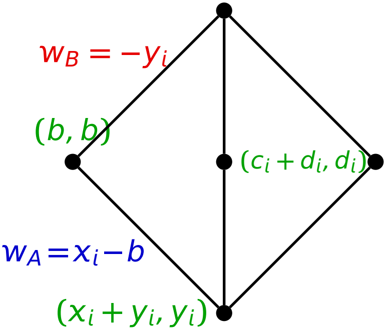
Painter wins by degeneracy.

(#tokens on a_2 is at least $d + \sum y_i$.)

$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

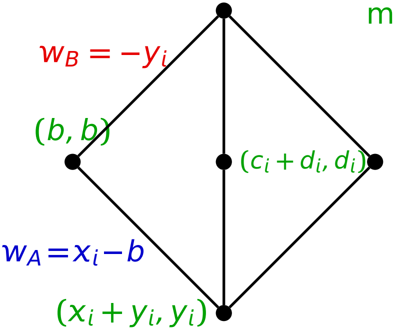
Painter uses strategy until a vertex is forced.

Case 2: b_1 is forced.



$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

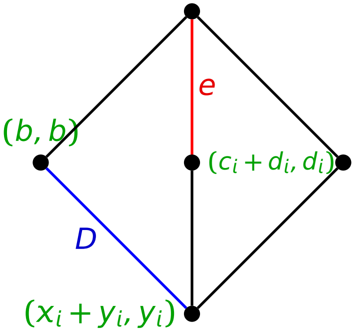


Case 2: b_1 is forced.

$\max\{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq b$

$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.



Case 2: b_1 is forced.

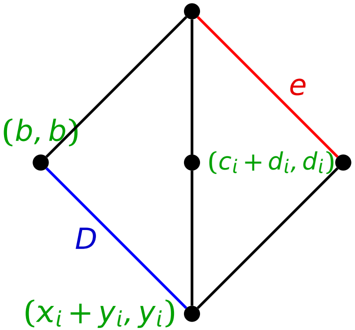
$\max\{w_A(e), w_B(e)\} \geq 0 \;\Rightarrow\; x_i \geq b$

$w_B(e) = c_2 - y_1, w_B(D) = -y_2$

$\Rightarrow\; c_2 \geq y_1 + y_2$

$(2m, m)$-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.



Case 2: b_1 is forced.

$\max\{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq b$

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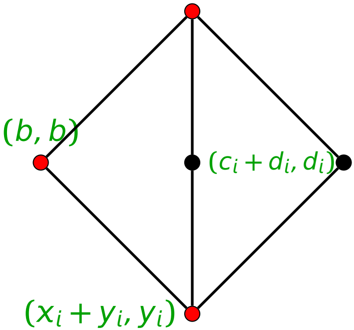

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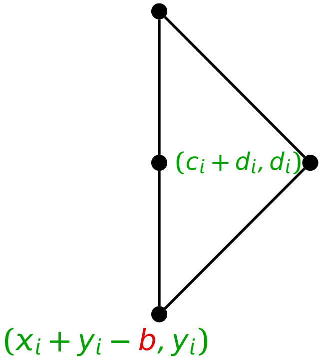

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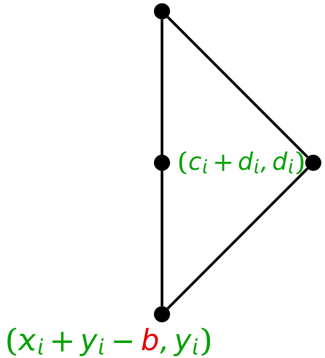

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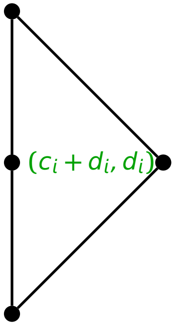

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$\therefore$ $(2,1)$-paintable $\Rightarrow$ $(2m,m)$-paintable.

Open Questions

Ques. Given 3-paint-critical G and $m > 1$, what is $\min t$ such that G is (t, m)-paintable?

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Thank You!