Homomorphism extension problem for Maltsev digraphs is easy

Alexandr Kazda

Vanderbilt University, Nashville

March 4, 2014

Homomorphism extension problem

- Let us fix our favorite digraph G.
- Ext(G): Given H and a partial mapping $f_0 : H \to G$, we want to know if we can extend f_0 to a digraph homomorphism.
- Equivalent way to state the same problem: Take G as a relational structure with one binary relation E...
- ...add for each $g \in G$ the unary relation $c_g = \{(g)\}$...
- ...and consider the constraint satisfaction problem with set of values $V(G)$ and relations $\{E\} \cup \{c_g, g \in G\}$.

Complexity of Ext(G)

- How hard is Ext(G)?
- Ext(K_3) is NP-complete.
- Ext(K_2) is in P (in fact in logspace).
- Classifying Ext(G) is an open problem.
- Dichotomy conjecture: If Ext(G) is not NP-complete, then it is in P.

Algebraic approach

- Consider all the mappings $q : V(G)^n \to V(G)$ such that q preserves $E(G)$ and $q(u, u, \ldots, u) = u$.
- This gives us the algebra of idempotent polymorphisms of G.
- The more idempotent polymorphisms G has, the less complicated $E(G)$ can be, and the easier $\mathrm{Ext}(G)$ becomes.
- It works the other way too: If G has no polymorphisms except for projections, then $\mathrm{Ext}(G)$ will be NP-complete.

Example: Maltsev polymorphism

- A polymorphism $p : V^3 \to V$ is Maltsev if for all vertices u and v we have

$$p(u, u, v) = p(v, u, u) = v.$$

- If p is a polymorphism of G, then $\mathrm{Ext}(G)$ can be solved in polynomial time.
- Can we do better?

Example: majority polymorphism

- A polymorphism $M : V^3 \to V$ is majority if for all vertices u and v we have

$$M(u, u, v) = M(u, v, u) = M(v, u, u) = u.$$

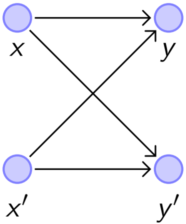
- If G has both Maltsev and majority, then there is a logarithmic space algorithm for $\mathrm{Ext}(G)$ (V. Dalmau, B. Larose, 2008 + O. Reingold, 2005).

Maltsev $\Rightarrow$ majority

- We will call a digraph G Maltsev resp. having a majority if $\operatorname{Pol} G$ contains a Maltsev resp. majority polymorphism.
- In general algebras, having a Maltsev operation does not imply having majority (consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$).
- However, we show that if a digraph is Maltsev then it does have a majority.
- From now on we will assume that G has a Maltsev operation p and has no sources or sinks.

Rectangularity

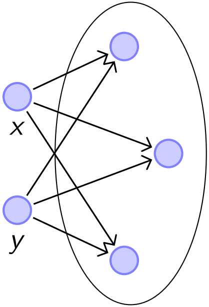
- Let x, y, x', y' be vertices of G and let $(x, y), (x', y'), (x', y) \in E$.



- Now apply the Maltsev polymorphism p and we get ...
- ... that $(x, y') \in E$ as well.
- We say that E is rectangular.

R^+ and R^-

- For v in V, we will denote by v^+ the vertex set $\{u \in V(G) : (v,u) \in E(G)\}$ by v^- the vertex set $\{u \in V(G) : (u,v) \in E(G)\}$.
- For u, v vertices of G, we write uR^+v if $u^+ = v^+$ and uR^-v if $u^- = v^-$.

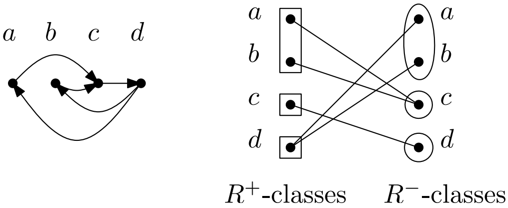


- In the picture, we have $x^+ = y^+$, therefore xR^+y.

R^+ and R^- are nice

- As E is rectangular, we obtain the following:
- The relations R^+ and R^- are equivalences on V.
- The mapping $\phi : E \mapsto E^+$ is a bijection from the set of equivalence classes of R^+ to the set of equivalence classes of R^-.

R^+ and R^- in a picture



The digraphs G^+ and G^-

- Given G, we define the digraph G^+ whose vertices are the equivalence classes of R^+ and $(U, V) \in E(G^+)$ iff there exist vertices $u \in U, v \in V$ with $(u, v) \in E(G)$.
- We define G^- similarly.
- G^+ and G^- are isomorphic.
- It turns out that if G is Maltsev then so is G^+.

Proof by induction

- Assume that G is the smallest Maltsev graph without a majority operation.
- If $|V(G^+)| = |V(G)|$ then G is a graph of a permutation and we win.
- Else we have a majority operation M^+ on G^+ and a corresponding M^- on G^-.

Extending the majority

- We have a majority operation M^+ on G^+ and M^- on G^- which we can extend to M on G by demanding that

$$[M(x,y,z)]_{R^+} = M^+([x]_{R^+},[y]_{R^+},[z]_{R^+})$$
$$[M(x,y,z)]_{R^-} = M^-([x]_{R^-},[y]_{R^-},[z]_{R^-})$$

- Examining R^+ and R^-, we discover that such an M always exists and is a majority polymorphism of G.

Bonus content

- What is a combinatorial description of Maltsev digraphs? They don't have an "N" subgraph (C. Carvalho, L. Egri, M. Jackson, T. Niven, 2011).
- Are there other nontrivial implications of the sort Maltsev implies majority that hold only for digraphs? Probably not. (J. Bulín, D. Delić, M. Jackson, T. Niven, 2013).
- Is there a CSP dichotomy for (symmetric) graphs? Yes (Hell-Nešetřil, 1990; Barto-Kozik, 2009).
- Is there a CSP dichotomy for oriented trees? We hope.

Thanks for your attention.