A problem of Erdős and Sós on 3-graphs

Jan Volec

University of Warwick

joint work with Roman Glebov and Daniel Král'

Motivation – Turán type questions

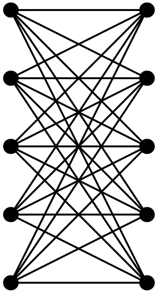
Mantel's Theorem (1907)

- n-vertex graph with no triangle – maximize number of edges?

Motivation – Turán type questions

Mantel's Theorem (1907)

- n-vertex graph with no triangle – maximize number of edges?

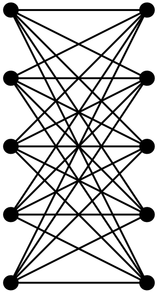


Complete bipartite graph has $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil \approx \frac{1}{2}\binom{n}{2}$ edges

Motivation – Turán type questions

Mantel's Theorem (1907)

- n-vertex graph with no triangle – maximize number of edges?



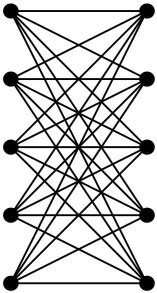
Complete bipartite graph has $\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil \approx \frac{1}{2}\binom{n}{2}$ edges

- Mantel's Theorem – this is the best possible

Motivation – Turán type questions

Mantel's Theorem (1907)

- n-vertex graph with no triangle – maximize number of edges?



Complete bipartite graph has $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil \approx \frac{1}{2}\binom{n}{2}$ edges

- Mantel's Theorem – this is the best possible
- The extremal graph is unique

Motivation – Turán type questions

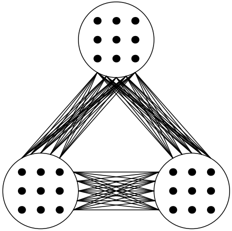
Turán's Theorem (1941)

- n-vertex graph with no K_k – maximize number of edges?

Motivation – Turán type questions

Turán's Theorem (1941)

- n-vertex graph with no K_k – maximize number of edges?

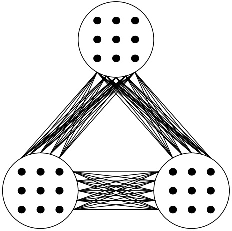


Complete balanced $(k-1)$-partite graph has $\approx \frac{k-2}{k-1}\binom{n}{2}$ edges

Motivation – Turán type questions

Turán's Theorem (1941)

- n-vertex graph with no K_k – maximize number of edges?



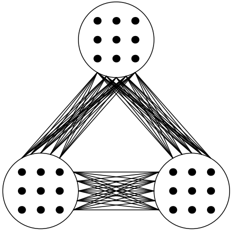
Complete balanced $(k-1)$-partite graph has $\approx \frac{k-2}{k-1}\binom{n}{2}$ edges

- Turán's Theorem – this is the best possible

Motivation – Turán type questions

Turán's Theorem (1941)

- n-vertex graph with no K_k – maximize number of edges?



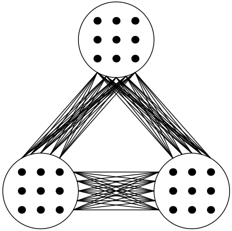
Complete balanced $(k-1)$-partite graph has $\approx \frac{k-2}{k-1}\binom{n}{2}$ edges

- Turán's Theorem – this is the best possible
- Again, the extremal graph is unique

Motivation – Turán type questions

Turán's Theorem (1941)

- n-vertex graph with no K_k – maximize number of edges?



Complete balanced $(k-1)$-partite graph has $\approx \frac{k-2}{k-1}\binom{n}{2}$ edges

- Turán's Theorem – this is the best possible
- Again, the extremal graph is unique

$\iff$ If density of G is larger than $\frac{k-2}{k-1}$, then G contains K_k

Graphs with positive density “everywhere”

- Suppose every subset of k vertices in G induces $\geq d\binom{k}{2}$ edges (for say $k \geq n/\log n$)

Graphs with positive density "everywhere"

- Suppose every subset of k vertices in G induces $\geq d\binom{k}{2}$ edges (for say $k \geq n/\log n$)
- What value of d guarantees a copy of triangle (or of K_k)?

Graphs with positive density "everywhere"

- Suppose every subset of k vertices in G induces $\geq d\binom{k}{2}$ edges (for say $k \geq n/\log n$)
- What value of d guarantees a copy of triangle (or of K_k)?
- Erdős and Sós ('82): if $d > 0$ and G large enough $\Rightarrow K_k \subseteq G$

Graphs with positive density "everywhere"

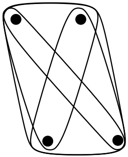
- Suppose every subset of k vertices in G induces $\geq d\binom{k}{2}$ edges (for say $k \geq n/\log n$)
- What value of d guarantees a copy of triangle (or of K_k)?
- Erdős and Sós ('82): if $d > 0$ and G large enough $\Rightarrow K_k \subseteq G$
- Erdős and Sós: Does it generalize to 3-uniform hypergraphs?

Graphs with positive density "everywhere"

- Suppose every subset of k vertices in G induces $\geq d\binom{k}{2}$ edges (for say $k \geq n/\log n$)
- What value of d guarantees a copy of triangle (or of K_k)?
- Erdős and Sós ('82): if $d > 0$ and G large enough $\Rightarrow K_k \subseteq G$
- Erdős and Sós: Does it generalize to 3-uniform hypergraphs?

Problem (Erdős and Sós: positive density $\Rightarrow K_4^3$)

For every $d > 0$, if G is 3-graph with sufficiently many vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^3?

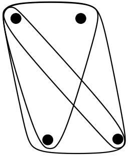


Graphs with positive density "everywhere"

- Suppose every subset of k vertices in G induces $\geq d\binom{k}{2}$ edges (for say $k \geq n/\log n$)
- What value of d guarantees a copy of triangle (or of K_k)?
- Erdős and Sós ('82): if $d > 0$ and G large enough $\Rightarrow K_k \subseteq G$
- Erdős and Sós: Does it generalize to 3-uniform hypergraphs?

Problem (Erdős and Sós: positive density $\Rightarrow K_4^-$)

For every $d > 0$, if G is 3-graph with sufficiently many vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-?



3-graphs with positive density "everywhere"

Problem (positive density $\Rightarrow K_4^-$)

For every $d > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-?

3-graphs with positive density "everywhere"

Problem (positive density $\Rightarrow K_4^-$)

For every $d > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-? **NO!**

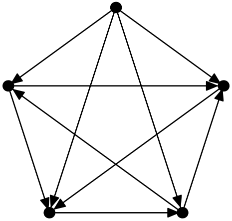
3-graphs with positive density "everywhere"

Problem (positive density $\Rightarrow K_4^-$)

For every $d > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-? **NO!**

Füredi (1983):

Random tournament $\Longrightarrow$ hyperedges $\leftrightarrow$ oriented triangles



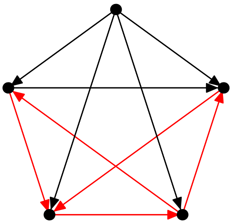
3-graphs with positive density "everywhere"

Problem (positive density $\Rightarrow K_4^-$)

For every $d > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-? **NO!**

Füredi (1983):

Random tournament $\Longrightarrow$ hyperedges $\leftrightarrow$ oriented triangles



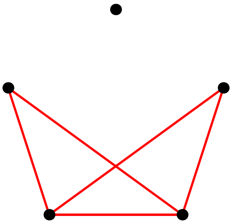
3-graphs with positive density "everywhere"

Problem (positive density $\Rightarrow K_4^-$)

For every $d > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-? **NO!**

Füredi (1983):

Random tournament $\Longrightarrow$ hyperedges $\leftrightarrow$ oriented triangles



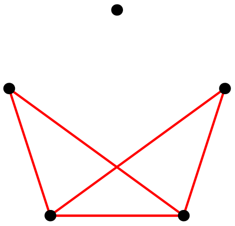
3-graphs with positive density "everywhere"

Problem (positive density $\Rightarrow K_4^-$)

For every $d > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-? **NO!**

Füredi (1983):

Random tournament $\Longrightarrow$ hyperedges $\leftrightarrow$ oriented triangles



- Density $1/4$ on every sufficiently large subset
- Any 4 vertices can span at most two edges

3-graphs with large density "everywhere"

Problem (large density "everywhere" $\Rightarrow K_4^-$)

For every $d > 1/4$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-?

3-graphs with large density "everywhere"

Problem (large density "everywhere" $\Rightarrow K_4^-$)

For every $d > 1/4$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-?

Theorem (Main result)

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq \delta n$ vertices induces $\geq (1/4 + \varepsilon)\binom{k}{3}$ edges, then G contains a copy of K_4^-.

3-graphs with large density "everywhere"

Problem (large density "everywhere" $\Rightarrow K_4^-$)

For every $d > 1/4$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-?

Theorem (Main result)

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq \delta n$ vertices induces $\geq (1/4 + \varepsilon)\binom{k}{3}$ edges, then G contains a copy of K_4^-.

Problem (large density "everywhere" $\Rightarrow K_4^3$)

For every $d > 1/2$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^3?

3-graphs with large density "everywhere"

Problem (large density "everywhere" $\Rightarrow K_4^-$)

For every $d > 1/4$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^-?

Theorem (Main result)

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq \delta n$ vertices induces $\geq (1/4 + \varepsilon)\binom{k}{3}$ edges, then G contains a copy of K_4^-.

Problem (large density "everywhere" $\Rightarrow K_4^3$)

For every $d > 1/2$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq n/\log n$ vertices induces $\geq d\binom{k}{3}$ edges, then G contains a copy of K_4^3?

The Turán question for both K_4^- and K_4^3 is still widely open

Conjectured value for K_4^- is $2/7$, for K_4^3 is $5/9$

Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq \delta n$ vertices induces $\geq (1/4 + \varepsilon)\binom{k}{3}$ edges, then G contains a copy of K_4^-.

Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges}$.

Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-

Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-
- for any 3-graph F, $p_k(F) := \mathrm{P}[\text{rand. } S \in \binom{V(H_k)}{|F|} \text{ induces } F]$

Overview of the proof

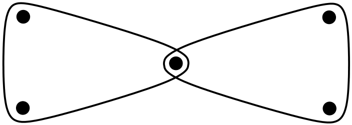
For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-
- for any 3-graph F, $p_k(F) := \mathrm{P}[\text{rand. } S \in \binom{V(H_k)}{|F|} \text{ induces } F]$
- WLOG, $p_k(F)$ converge for every F, let $q(F)$ be the limit

Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-
- for any 3-graph F, $p_k(F) := \mathrm{P}[\text{rand. } S \in \binom{V(H_k)}{|F|} \text{ induces } F]$
- WLOG, $p_k(F)$ converge for every F, let $q(F)$ be the limit
- Claim: convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$



B – butterfly

Overview of the proof

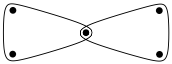
For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-
- for any 3-graph F, $p_k(F) := \mathrm{P}[\text{rand. } S \in \binom{V(H_k)}{|F|} \text{ induces } F]$
- WLOG, $p_k(F)$ converge for every F, let $q(F)$ be the limit
- Claim: convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$
- delete each edge in H_k randomly with small probability $\to H_k'$ H_k' satisfies $\mathcal{P}(0, \frac{1}{k})$ and (H_k') converges to $q' \implies q'(B) = 0$

Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-
- for any 3-graph F, $p_k(F) := \mathrm{P}[\text{rand. } S \in \binom{V(H_k)}{|F|} \text{ induces } F]$
- WLOG, $p_k(F)$ converge for every F, let $q(F)$ be the limit
- Claim: convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$
- delete each edge in H_k randomly with small probability $\to H_k'$ H_k' satisfies $\mathcal{P}(0, \frac{1}{k})$ and (H_k') converges to $q' \implies q'(B) = 0$
- $\implies q(F) = 0$ for any F that contain B as subgraph



Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-
- for any 3-graph F, $p_k(F) := \mathrm{P}[\text{rand. } S \in \binom{V(H_k)}{|F|} \text{ induces } F]$
- WLOG, $p_k(F)$ converge for every F, let $q(F)$ be the limit
- Claim: convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$
- delete each edge in H_k randomly with small probability $\rightarrow H'_k$ H'_k satisfies $\mathcal{P}(0, \frac{1}{k})$ and (H'_k) converges to $q' \implies q'(B) = 0$
- $\implies q(F) = 0$ for any F that contain B as subgraph
- $\implies$ edge densities of H_n tend to zero – a contradiction

Overview of the proof

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices satisfying $\mathcal{P}(\varepsilon, \delta)$ then G contains a copy of K_4^-, where G satisfies $\mathcal{P}(\varepsilon, \delta) \iff \forall A \subseteq V(G), |A| \geq \delta n, A \text{ induces} \geq (1/4 + \varepsilon)\binom{|A|}{3} \text{ edges.}$

- Suppose it is false $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall k$ there is H_k such that $|H_k| \geq k$, H_k satisfies $\mathcal{P}(\varepsilon_0, \frac{1}{k})$, and does not contain K_4^-
- for any 3-graph F, $p_k(F) := \mathrm{P}[\text{rand. } S \in \binom{V(H_k)}{|F|} \text{ induces } F]$
- WLOG, $p_k(F)$ converge for every F, let $q(F)$ be the limit
- Claim: convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$
- delete each edge in H_k randomly with small probability $\rightarrow H'_k$ H'_k satisfies $\mathcal{P}(0, \frac{1}{k})$ and (H'_k) converges to $q' \implies q'(B) = 0$
- $\implies q(F) = 0$ for any F that contain B as subgraph
- $\implies$ edge densities of H_n tend to zero – a contradiction

Flag Algebras (graph case)

- framework developed by Razborov (2010)

Flag Algebras (graph case)

- framework developed by Razborov (2010)
- related to theory of graph limits – Lovász, Szegedy (2006) and Borgs, Chayes, Lovász, Sós, Vesztergombi (2008)

Flag Algebras (graph case)

- framework developed by Razborov (2010)
- related to theory of graph limits – Lovász, Szegedy (2006) and Borgs, Chayes, Lovász, Sós, Vesztergombi (2008)
- consider sequence of graphs $G_1, G_2, \dots$ (with $|G_k| \to \infty$)

Flag Algebras (graph case)

- framework developed by Razborov (2010)
- related to theory of graph limits – Lovász, Szegedy (2006) and Borgs, Chayes, Lovász, Sós, Vesztergombi (2008)
- consider sequence of graphs $G_1, G_2, \ldots$ (with $|G_k| \to \infty$)
- $p_k(F) :=$ probability that random $|F|$ vertices of G_k induces F

Flag Algebras (graph case)

- framework developed by Razborov (2010)
- related to theory of graph limits – Lovász, Szegedy (2006) and Borgs, Chayes, Lovász, Sós, Vesztergombi (2008)
- consider sequence of graphs $G_1, G_2, \dots$ (with $|G_k| \to \infty$)
- $p_k(F) :=$ probability that random $|F|$ vertices of G_k induces F
- always has a subsequence s.t. values $p_k(F)$ converge for all F

Flag Algebras (graph case)

- framework developed by Razborov (2010)
- related to theory of graph limits – Lovász, Szegedy (2006) and Borgs, Chayes, Lovász, Sós, Vesztergombi (2008)
- consider sequence of graphs $G_1, G_2, \ldots$ (with $|G_k| \to \infty$)
- $p_k(F) :=$ probability that random $|F|$ vertices of G_k induces F
- always has a subsequence s.t. values $p_k(F)$ converge for all F
- limit object – function q from graphs to $[0, 1]$

Flag Algebras (graph case)

- framework developed by Razborov (2010)
- related to theory of graph limits – Lovász, Szegedy (2006) and Borgs, Chayes, Lovász, Sós, Vesztergombi (2008)
- consider sequence of graphs $G_1, G_2, \ldots$ (with $|G_k| \to \infty$)
- $p_k(F) :=$ probability that random $|F|$ vertices of G_k induces F
- always has a subsequence s.t. values $p_k(F)$ converge for all F
- limit object – function q from graphs to $[0, 1]$
- q yields homomorphism from algebra of formal linear combinations of graphs to reals

Flag Algebras – basic properties of q

- linear extension of q:

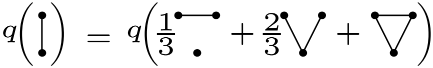
$$q(\alpha_1 F_1 + \alpha_2 F_2) := \alpha_1 q(F_1) + \alpha_2 q(F_2)$$

Flag Algebras – basic properties of q

- linear extension of q:

$$q(\alpha_1 F_1 + \alpha_2 F_2) := \alpha_1 q(F_1) + \alpha_2 q(F_2)$$

- lifting up in q:

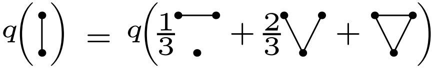


Flag Algebras – basic properties of q

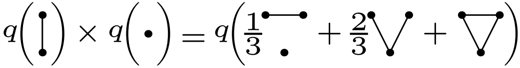
- linear extension of q:

$$q(\alpha_1 F_1 + \alpha_2 F_2) := \alpha_1 q(F_1) + \alpha_2 q(F_2)$$

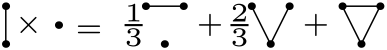
- lifting up in q:



- product of graphs in q:



$\Longrightarrow$ define

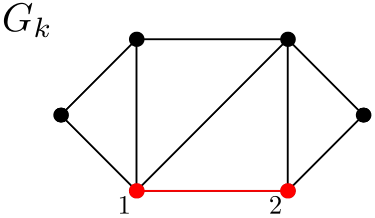


Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's

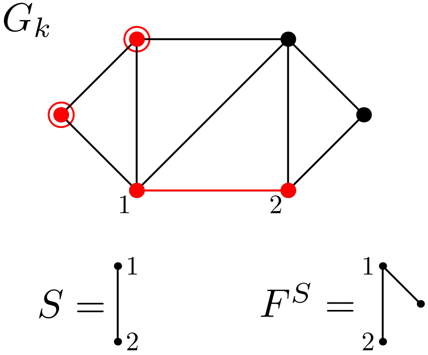
Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \to$ function p_k^C



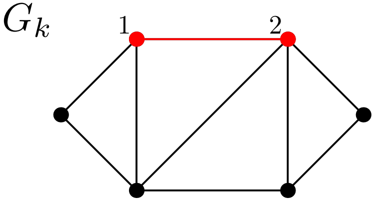
Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \to$ function p_k^C



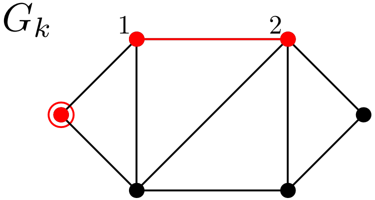
Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of S → function p_k^C



Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \to$ function p_k^C



Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \rightarrow$ function p_k^C
- a random copy $\mathbf{C}$ of S in $G_k \rightarrow$ a random function $\mathbf{p}_k^S$

Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \to$ function p_k^C
- a random copy $\mathbf{C}$ of S in $G_k \to$ a random function $\mathbf{p}_k^S$
- rand. functions $\mathbf{p}_k^S$ weakly converge to a random function $\mathbf{q}^S$

Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \rightarrow$ function p_k^C
- a random copy $\mathbf{C}$ of S in $G_k \rightarrow$ a random function $\mathbf{p}_k^S$
- rand. functions $\mathbf{p}_k^S$ weakly converge to a random function $\mathbf{q}^S$
- furthermore, q uniquely determines $\mathbf{q}^S$ for every S

Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \to$ function p_k^C
- a random copy $\mathbf{C}$ of S in $G_k \to$ a random function $\mathbf{p}_k^S$
- rand. functions $\mathbf{p}_k^S$ weakly converge to a random function $\mathbf{q}^S$
- furthermore, q uniquely determines $\mathbf{q}^S$ for every S
- Averaging argument:

$$q(K_2) = \mathbb{E}_v \mathbf{q}^v \left(K_2^v\right)$$

Flag Algebras – rooted homomorphisms

- let S be a graph with $p_k(S) > 0$ for all sufficiently large k's
- for a fixed copy C of S in G_k, consider probabilities $p_k^C(F^S)$
  F^S – a graph with fixed embedding of $S \rightarrow$ function p_k^C
- a random copy $\mathbf{C}$ of S in $G_k \rightarrow$ a random function $\mathbf{p}_k^S$
- rand. functions $\mathbf{p}_k^S$ weakly converge to a random function $\mathbf{q}^S$
- furthermore, q uniquely determines $\mathbf{q}^S$ for every S
- Averaging argument:

$$q(K_2) = \mathbb{E}_v \mathbf{q}^v \left(K_2^v\right)$$

- Cauchy-Schwarz inequality:

$$\mathbb{E}_S \mathbf{q}^S \left(F \times F\right) \geq 0$$

Flag Algebras – Mantel's Theorem

If (G_n) conv. sequence of triangle-free graphs, then $q(e) \leq \frac{1}{2}$.

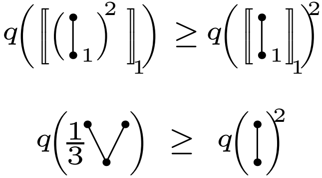
Flag Algebras – Mantel's Theorem

If (G_n) conv. sequence of triangle-free graphs, then $q(e) \leq \frac{1}{2}$.

$$q\left(\left[\!\left[\left(\overset{\bullet}{\underset{\bullet_{1}}{|}}\right)^2\right]\!\right]_1\right) \geq q\left(\left[\!\left[\overset{\bullet}{\underset{\bullet_{1}}{|}}\right]\!\right]_1\right)^2$$

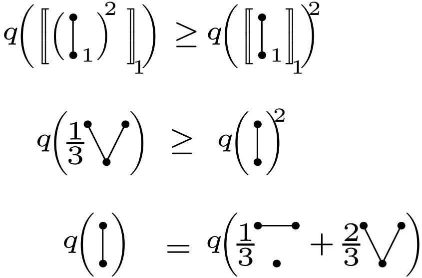
Flag Algebras – Mantel's Theorem

If (G_n) conv. sequence of triangle-free graphs, then $q(e) \leq \frac{1}{2}$.



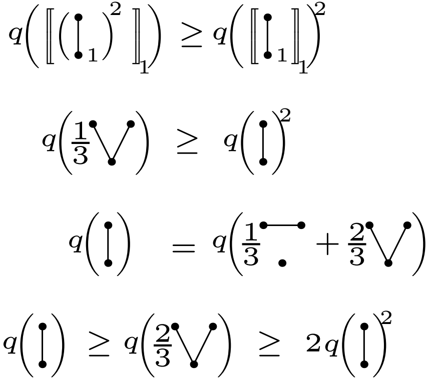
Flag Algebras – Mantel's Theorem

If (G_n) conv. sequence of triangle-free graphs, then $q(e) \leq \frac{1}{2}$.



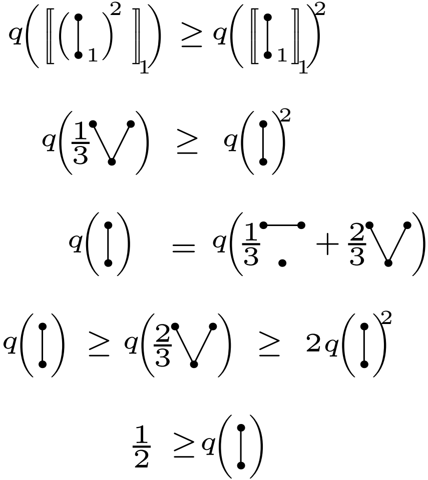
Flag Algebras – Mantel's Theorem

If (G_n) conv. sequence of triangle-free graphs, then $q(e) \leq \frac{1}{2}$.



Flag Algebras – Mantel's Theorem

If (G_n) conv. sequence of triangle-free graphs, then $q(e) \leq \frac{1}{2}$.



Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A$ induces $\geq 1/4\binom{|A|}{3}$ edges

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$

Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A \text{ induces} \geq 1/4\binom{|A|}{3} \text{ edges}$

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$
- If U is "small" $\Rightarrow$ relative density of U is also "small"

Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A \text{ induces} \geq 1/4\binom{|A|}{3} \text{ edges}$

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$
- If U is "small" $\Rightarrow$ relative density of U is also "small"
- If U is "large" $\Rightarrow$ relative density of U is $\geq 1/4$

Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A \text{ induces} \geq 1/4\binom{|A|}{3} \text{ edges}$

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$
- If U is "small" $\Rightarrow$ relative density of U is also "small"
- If U is "large" $\Rightarrow$ relative density of U is $\geq 1/4$
- Normalized number of all triplets in U is $q^{v,w}\left(K_3^{v,w}\right)^3$

Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A \text{ induces} \geq 1/4\binom{|A|}{3} \text{ edges}$

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$
- If U is "small" $\Rightarrow$ relative density of U is also "small"
- If U is "large" $\Rightarrow$ relative density of U is $\geq 1/4$
- Normalized number of all triplets in U is $q^{v,w}\left(K_3^{v,w}\right)^3$
- analogously, we express normalized number of edges in U

Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A \text{ induces} \geq 1/4\binom{|A|}{3} \text{ edges}$

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$
- If U is "small" $\Rightarrow$ relative density of U is also "small"
- If U is "large" $\Rightarrow$ relative density of U is $\geq 1/4$
- Normalized number of all triplets in U is $q^{v,w}\left(K_3^{v,w}\right)^3$
- analogously, we express normalized number of edges in U
- $\Longrightarrow 4 \cdot \mathrm{P}[T \subseteq U \wedge T \in E(G_k)] - \mathrm{P}[T \subseteq U] \geq 0$

Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A \text{ induces} \geq 1/4\binom{|A|}{3} \text{ edges}$

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$
- If U is "small" $\Rightarrow$ relative density of U is also "small"
- If U is "large" $\Rightarrow$ relative density of U is $\geq 1/4$
- Normalized number of all triplets in U is $q^{v,w}\left(K_3^{v,w}\right)^3$
- analogously, we express normalized number of edges in U
- $\Longrightarrow 4 \cdot \mathrm{P}[T \subseteq U \wedge T \in E(G_k)] - \mathrm{P}[T \subseteq U] \geq 0$
- Similarly an inequality for $U' := \{u : uvw \notin E(G_k)\}$

Flag Algebras – how to express large density "everywhere"?

$\forall A \subseteq V(G_k), |A| \geq \frac{1}{k}|V(G_k)|, A \text{ induces} \geq 1/4\binom{|A|}{3} \text{ edges}$

- Fix two vertices v, w in G_k and $U := \{u : uvw \in E(G_k)\}$
- If U is "small" $\Rightarrow$ relative density of U is also "small"
- If U is "large" $\Rightarrow$ relative density of U is $\geq 1/4$
- Normalized number of all triplets in U is $q^{v,w}\left(K_3^{v,w}\right)^3$
- analogously, we express normalized number of edges in U
- $\Longrightarrow 4 \cdot \mathrm{P}[T \subseteq U \wedge T \in E(G_k)] - \mathrm{P}[T \subseteq U] \geq 0$
- Similarly an inequality for $U' := \{u : uvw \notin E(G_k)\}$
- Holds for any choice of v, w. Averaging $\longrightarrow$ inequality for q

Finishing the proof

Claim: Convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$

Finishing the proof

Claim: Convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$

- appopriate combination of the two inequalities and many C-S's

Finishing the proof

Claim: Convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$

- appopriate combination of the two inequalities and many C-S's
- Computer-assisted proof

Finishing the proof

Claim: Convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$

- appopriate combination of the two inequalities and many C-S's
- Computer-assisted proof
- the final size of the output – more than 10 megabytes :-(

Finishing the proof

Claim: Convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$

- appopriate combination of the two inequalities and many C-S's
- Computer-assisted proof
- the final size of the output – more than 10 megabytes :-(
- in fact, $q(F) = 0$ for any F not in tournaments with $|F| \leq 7$

Finishing the proof

Claim: Convergent (G_k) s.t. G_k satisfies $\mathcal{P}(0, \frac{1}{k}) \Rightarrow q(B) = 0$

- appopriate combination of the two inequalities and many C-S's
- Computer-assisted proof
- the final size of the output – more than 10 megabytes :-(
- in fact, $q(F) = 0$ for any F not in tournaments with $|F| \leq 7$

Theorem (Main result)

For every $\varepsilon > 0$ there is $\delta > 0, n_0$ s.t. if G is 3-graph on $n \geq n_0$ vertices and every subset of $k \geq \delta n$ vertices induces $\geq (1/4 + \varepsilon)\binom{k}{3}$ edges, then G contains a copy of K_4^-.

Related problems and future work

Theorem (Falgas-Ravry, Pikhurko, Vaughan 2013+)

For every $\varepsilon > 0$ *there is* n_0 *s.t. if* G *is* 3*-graph on* $n \geq n_0$ *vertices and any two vertices are in* $\geq (\frac{1}{4} + \varepsilon)n$ *edges, then* $K_4^- \subseteq G$*.*

Related problems and future work

Theorem (Falgas-Ravry, Pikhurko, Vaughan 2013+)

For every $\varepsilon > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and any two vertices are in $\geq (\frac{1}{4} + \varepsilon)n$ edges, then $K_4^- \subseteq G$.

- *The extremal example (as the limit) is unique.*
- *If co-degrees tend to $\frac{1}{4} - \varepsilon$, the limit is close to the extremal.*

Related problems and future work

Theorem (Falgas-Ravry, Pikhurko, Vaughan 2013+)

For every $\varepsilon > 0$ *there is* n_0 *s.t. if* G *is* 3*-graph on* $n \geq n_0$ *vertices and any two vertices are in* $\geq (\frac{1}{4} + \varepsilon)n$ *edges, then* $K_4^- \subseteq G$*.*

- *The extremal example (as the limit) is unique.*
- *If co-degrees tend to* $\frac{1}{4} - \varepsilon$*, the limit is close to the extremal.*

Open problems:

- The uniqueness and stability for our result?

Related problems and future work

Theorem (Falgas-Ravry, Pikhurko, Vaughan 2013+)

For every $\varepsilon > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and any two vertices are in $\geq (\frac{1}{4} + \varepsilon)n$ edges, then $K_4^- \subseteq G$.

- *The extremal example (as the limit) is unique.*
- *If co-degrees tend to $\frac{1}{4} - \varepsilon$, the limit is close to the extremal.*

Open problems:

- The uniqueness and stability for our result?
- Is $\frac{1}{2}$ the correct value for the K_4 problem?

Related problems and future work

Theorem (Falgas-Ravry, Pikhurko, Vaughan 2013+)

For every $\varepsilon > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and any two vertices are in $\geq (\frac{1}{4} + \varepsilon)n$ edges, then $K_4^- \subseteq G$.

- *The extremal example (as the limit) is unique.*
- *If co-degrees tend to $\frac{1}{4} - \varepsilon$, the limit is close to the extremal.*

Open problems:

- The uniqueness and stability for our result?
- Is $\frac{1}{2}$ the correct value for the K_4 problem?
- The co-degree problem for K_4 (conjecture is again $\frac{1}{2}$)

Related problems and future work

Theorem (Falgas-Ravry, Pikhurko, Vaughan 2013+)

For every $\varepsilon > 0$ there is n_0 s.t. if G is 3-graph on $n \geq n_0$ vertices and any two vertices are in $\geq (\frac{1}{4} + \varepsilon)n$ edges, then $K_4^- \subseteq G$.

- *The extremal example (as the limit) is unique.*
- *If co-degrees tend to $\frac{1}{4} - \varepsilon$, the limit is close to the extremal.*

Open problems:

- The uniqueness and stability for our result?
- Is $\frac{1}{2}$ the correct value for the K_4 problem?
- The co-degree problem for K_4 (conjecture is again $\frac{1}{2}$)
- Analogous problems for r-uniform hypergraphs ($r > 3$)

Thank you for your attention!