The Ramsey number of the clique and the hypercube

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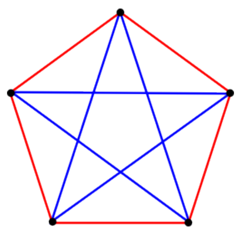
Ramsey theory, basics

- $r(n,s)$: smallest N such that any colouring of the edges of K_N with blue and red contains a blue clique on n vertices or a red clique on s vertices
- $r(3,3) = 6$ 
- Ramsey, 1930: $r(n,s) < \infty$
- Celebrated problem: determining/estimating $r(n,n)$
- Erdős and Szekerés, 1935: $r(n,n) \leq 2^{2n}$
- Erdős, 1947: $2^{n/2} \leq r(n,n)$

Graph Ramsey theory

- ▷ $r(G, H)$: smallest N such that any colouring of the edges of K_N with blue and red contains a blue G or a red H
- ▷ Existence: trivial, from $r(G, H) \leq r(|G|, |H|) < \infty$

Motivation: test the limits of current methods and develop a new ones that could be (possibly) used to improve bounds on $r(n, n)$

Examples:

- ▷ Gerencsér, Gyárfás, 1967: $r(P_n, P_m) = n + \lfloor m/2 \rfloor - 1$ for $n \geq m \geq 2$
- ▷ Chvátal, 1977: $r(T_n, K_m) = (m-1)(n-1) + 1$ for $m \geq 2$, any tree T_n

Simple Lower Bound

Theorem 1 (Chvátal, Harary, 1972)**.** *If G is connected, then*

$$r(G,H) \geq (\chi(H)-1)(|G|-1)+1.$$

Proof.

- ▷ take $\chi(H)-1$ disjoint BLUE cliques of size $|G|-1$;
- ▷ colour by RED all pairs between two red cliques;
- ▷ G does not fit to one BLUE clique of size $|G|-1$;
- ▷ RED subgraph has chromatic number $\chi(H)-1$ and cannot contain H.

□

Simple Lower Bound, cont.

- $\sigma(H)$: the minimum size of a colour class taken over all $\chi(H)$-colourings of H
- $\sigma(K_m) = 1$, $\sigma(P_m) = \lfloor m/2 \rfloor$, $\sigma(C_{2m-1}) = 1$, $\sigma(C_{2m}) = m$

Theorem 2 (Burr, 1980)**.** *If G is connected and $|G| \geq \sigma(H)$, then*

$$r(G, H) \geq (\chi(H) - 1)(|G| - 1) + \sigma(H).$$

Ramsey Goodness

- G is $\boldsymbol{H}$**-good** if $r(G, H) = (\chi(H) - 1)(|G| - 1) + \sigma(H)$

Examples:

- Chvátal, 1977: all trees are K_m-good for $m \geq 2$
- Bondy, Erdős, 1973: C_n is K_m-good for $n \geq m^2 - 2$
  - Nikiforov, 2008: C_n is K_m-good for $n \geq 4m + 2$
  - Conjecture (Erdős): C_n is K_m-good for $n \geq m$
- Interesting problem: behaviour of $r(K_m, C_n)$ for fixed $m \geq 3$

Burr–Erdős conjectures

Burr, Erdős: Which graph parameters control the growth of $r(G, H)$?

Conjecture 1 (Burr, 1981)**.** *Fix a graph* H *and* $\Delta \in \mathbb{N}$*. Then any large connected graph* G *with* $\Delta(G) \leq \Delta$ *is* H*-good:*

$$r(G, H) = (\chi(H) - 1)(|G| - 1) + \sigma(H).$$

G is Δ-degenerate: every subgraph of G has a vertex of degree $\leq \Delta$

Conjecture 2 (Burr and Erdős, 1983)**.** *Fix* $m \geq 3$ *and* $\Delta \geq 1$*. Then every large connected* Δ*-degenerate graph is* K_m*-good.*

Burr–Erdős conjectures, cont.

- ▷ Burr, Erdős: gave a list of test graphs that should be K_m-good: wheels $K_1 + C_n$, certain subdivisions of K_n, $K_1 + C_n^k$, $\ldots$, hypercubes.

- ▷ Nikiforov and Rousseau, 2009: resolved (positively) all but one of these questions.

- ▷ remaining question (modest version): Is Q_n K_3-good for large n?

Conjectures 1 and 2 are false

- ▷ Brandt, 1996: there exists a 168-regular graph that is not K_3-good
- ▷ Nikiforov and Rousseau, 2009: almost all 100-regular graphs are not K_3-good
- ▷ counterexamples have good expansion properties
- ▷ what if we limit expansion?
- ▷ bandwidth $\mathrm{bw}(G)$: smallest k such that $V(G)$ has ordering $v_1, v_2, \ldots, v_{|G|}$:
  if $v_i v_j \in E(G)$ then $|i - j| \leq k$.

Conjecture 1 'rescued'

Theorem 3 (Allen, Brightwell, S., 2013)**.** *Fix* $k \in \mathbb{N}$ *and a graph* H*. Then any sufficiently large connected graph* G *with* $\mathrm{bw}(G) \leq k$ *is* H*-good:*

$$r(G, H) = (\chi(H) - 1)(|G| - 1) + \sigma(H).$$

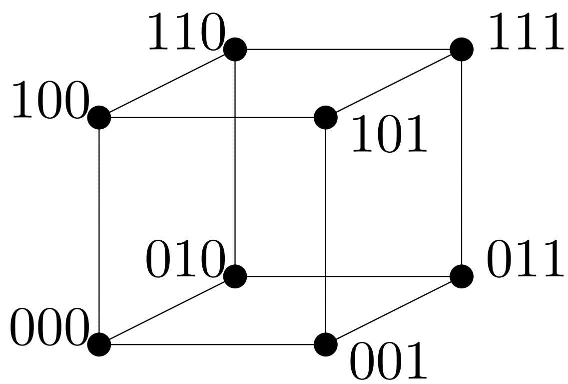
Note: k and H can grow with $|G|$ 'reasonably fast'

Theorem 4 (Allen, Brightwell, S., 2013)**.** *Fix* $\Delta \in \mathbb{N}$ *and a graph* H*. Then any sufficiently large connected graph* G *with* $\Delta(G) \leq \Delta$ *and* $\mathrm{bw}(G) = o(|G|)$ *is* H*-good.*

Hypercube Q_n

$V(Q_n)$: all sequences $\mathbf{x} = x_1 x_2 \ldots x_n$ with entries from $\{0, 1\}$

$E(Q_n)$: $\mathbf{xy}$ is an edge if $\mathbf{x}$ and $\mathbf{y}$ differ in exactly one position



hypercube Q_3

Properties: n-regular on 2^n vertices, bipartite, $\mathrm{bw}(Q_n) \sim 2^n/\sqrt{n}$

Are hypercubes good?

We would like to see whether, for H fixed and n large,

$$r(Q_n, H) = (\chi(H) - 1)(2^n - 1) + \sigma(H).$$

For $H = K_3$, is

$$r(Q_n, K_3) = 2^{n+1} - 1?$$

Theorems 3 and 4 <u>do not</u> apply: $\mathrm{bw}(Q_n)$ and $\Delta(Q_n)$ grow too fast.

$$2^{n+1} - 1 \leq R(Q_n, K_3) \leq (n+1)(2^n - 1) + 1$$

In any 2-colouring of K_N with $N = (n+1)(2^n - 1) + 1$ and no red K_3

- ▷ the red neighbours of any vertex form a blue clique
- ▷ hence the red degree of any vertex $\leq 2^n - 1$ (or blue Q_n)
- ▷ take any ordering $v_1, \ldots, v_{2^n}$ of $V(Q_n)$, embed greedily
- ▷ v_i has at most n neighbours in $v_1, \ldots, v_{i-1}$ and they have $\leq n(2^n - 1)$ red neighbours in K_N
- ▷ v_i has $\geq N - n(2^n - 1) - (i - 1) > 0$ available images

We only used that Q_n is n-regular.

Results

- ▷ Conlon, Fox, Lee, Sudakov (2012+)

$$R(Q_n, K_3) \leq 7200 \cdot 2^n \text{ for } n \geq 6\text{, and}$$

$$R(Q_n, H) \leq c_H 2^n \text{ for } H \text{ fixed and } n \text{ large.}$$

- ▷ Fiz Pontiveros, Griffiths, Morris, Saxton, S. (2013+)

$$R(Q_n, K_3) = (1 + o(1))2^{n+1} \text{as } n \to \infty.$$

- ▷ Fiz Pontiveros, Griffiths, Morris, Saxton, S. (2013+)

Q_n is H-good for any fixed H and n large.

Some ideas from $R(Q_n, K_3) \leq 7000 \cdot 2^n$

initial subcube of co-dimension d: for $\mathbf{x} = (x_1, \ldots, x_d) \in \{0,1\}^d$, let

$$Q_{\mathbf{x}} = \{\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \{0,1\}^n : y_i = x_i \text{ for each } 1 \leq i \leq d\}$$

Note: every vertex $v \in V(Q_{\mathbf{x}})$ has exactly d neighbours in $V(Q_n) \setminus V(Q_{\mathbf{x}})$

Observations: for $\mathbf{x} \in \{0,1\}^d$ and $\mathbf{z} \in \{0,1\}^{d'}$, $v \in Q_{\mathbf{x}}$, and $w \in Q_{\mathbf{z}}$,

- ▷ if $Q_{\mathbf{x}} \cap Q_{\mathbf{z}} \neq \emptyset$, then $Q_{\mathbf{x}} \subseteq Q_{\mathbf{z}}$ or $Q_{\mathbf{z}} \subseteq Q_{\mathbf{x}}$.
- ▷ $Q_{\mathbf{x}} \cap Q_{\mathbf{z}} = \emptyset$ iff $\mathbf{x}$ and $\mathbf{z}$ differ in at least one of the first $\min\{d, d'\}$ coordinates.
- ▷ when $Q_{\mathbf{x}} \cap Q_{\mathbf{z}} = \emptyset$, $vw \in E(Q_n)$ iff $\mathbf{x}$ and $\mathbf{z}$ differ in precisely one of the first $\min\{d, d'\}$ coordinates. We call such $Q_{\mathbf{x}}$ and $Q_{\mathbf{z}}$ adjacent.

Find

- ▷ decomposition of $Q_n = Q^1 \cup Q^2 \cup \cdots \cup Q^\ell$ into initial subcubes, Q^i of co-dimension d_i, $d_1 \leq d_2 \leq \ldots d_\ell$,
- ▷ a collection of BLUE cliques $T^1, T^2 \ldots, T^\ell$ in K_N,

such that

- ▷ $|T^i| = 2|Q^i| = 2 \cdot 2^{n-d_i}$,
- ▷ if Q^i and Q^j are adjacent, $i < j$, then each vertex of T^j has at most $|T^i|/4d_i$ RED neighbours in T_i.

Embed $Q^\ell \to T^\ell, Q^{\ell-1} \to T^{\ell-1} \ldots Q^1 \to T^1$ sequentially and greedily. Constant 7200 comes from the construction of this setup.

Some ideas from $R(Q_n, K_3) = (1 + o(1))2^{n+1}$

Take 2-coloured K_N with $N = (1 + 2\epsilon)2^{n+1}$ with no RED K_3.

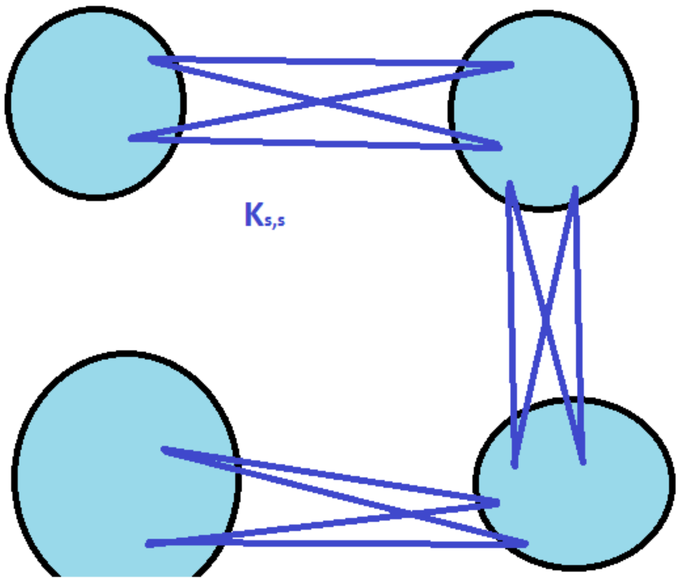
Case 1: there exists C, $|C| > (1 + \epsilon)2^n$ such that every vertex $v \in C$ has at most $2^n / \log \log \log n$ RED neighbours in C.

Then we can find a setup similar to the previous proof and embed Q_n to C.

Case 2: there exist a collection of (m, s)-snakes covering $> (1 + \epsilon)2^n$ with small RED degrees between them

(m, s)-snake: a collection of BLUE cliques of size m connected by BLUE $K_{s,s}$

(m, s)-snakes contain large powers of the path



Motivation: any ℓ vertex graph with bandwidth s is a subgraph of P_ℓ^s, we can embed P_ℓ^s to an (m, s)-snake S if $|S| \geq \ell$.

In particular, $Q_n \subset P_\ell^s$, where $\ell \geq 2^n$ and $s \geq 2^n/n^{1/3} > \mathrm{bw}(Q_n)$.

Precise statement

Fix n large, let G be a two-coloured complete graph with no blue triangles, and with $2^n \le v(G) \le 2^{n+2}$. Then there exists a partition of $V(G)$ into sets $C \cup S_1 \cup \cdots \cup S_r$, for some $r \ge 0$, such that the following conditions hold:

(a) $e\big(G_B[C]\big) \le \frac{2^n|C|}{\log\log n}$

and, for every $i \in [r]$, there exists $n^{-1/3} \le s_i \cdot 2^{-n} \le n^{-1/4}$ such that

(b) $G_R[S_i]$ contains a spanning (m, s_i)-snake S_i, where $m = \frac{2^{n-1}}{\log\log n}$.

(c) $|N_B(v) \cap S_i| \le \frac{s_i}{\log\log n}$ for every $v \in S_{i+1} \cup \cdots \cup S_r$.

Questions ....

- $\triangleright$ What additional conditions imply that Δ-degenerate graph G is H-good ?
- $\triangleright$ Is it true that $r(Q_n, Q_n) = O(2^n)$? (Fox, Sudakov 2009: $\leq n2^{2n+5}$)
- $\triangleright$ Is it true that $r(Q_n, Q_m) = O(2^n)$ for m growing with n ?
- $\triangleright$ For large G with $\Delta(G) \leq \Delta$ and $\mathrm{bw}(G) = o(|G|)$, it appears that

$$r(G, G) \leq (\chi(G) + \alpha)|G| + \beta.$$

What are α and β?

- ▷ Is it true that for all graphs G with maximum degree at most Δ, we have

$$r(G,G) \leq 2^{c\Delta}|G|?$$

Conlon, Fox, Sudakov 2009: $\leq 2^{c\Delta \log \Delta}|G|$

- ▷ Is it true that for all Δ-degenerate graphs G, we have

$$r(G,G) \leq c(\Delta)|G|?$$

Kostochka, Sudakov 2003: $\leq |G|^{1+o(1)}$