Improper Coloring, a relaxation of Steinberg Conjecture and (i, j, k)-coloring of graphs

André Raspaud

LaBRI
Université Bordeaux I
France

Proper k-coloring

Proper k-coloring

Let G be a graph and k ($k \geq 1$) an integer.
A proper k-coloring of G is a mapping $\phi : V(G) \longrightarrow \{1, \cdots, k\}$ such that:

- for every edge xy, $\phi(x) \neq \phi(y)$

In other words, a k-coloring of G is a partition $V_1, V_2, \cdots, V_k$ of $V(G)$ such that V_i is an independent set for every i, i.e., the subgraph induced by V_i has maximum degree zero.

d-improper k-coloring

Burr and Jacobson (1985), Cowen, Cowen, and Woodall (1986), Harary and Jones (1985).

d-improper k-coloring

Let G be a graph and k, d $(k, d \geq 1)$ integers.
A d-improper k-coloring of G is a mapping $\psi : V(G) \longrightarrow \{1, \cdots, k\}$ such that :

- $\forall i, 1 \leq i \leq k$, $G[i]$ has a maximum degree at most d
- $G[i]$ is the subgraph induced by color i.

Every vertex v has at most d neighbors receiving the same color as v.

- a $(0, 0, 0, 0)$-coloring is a proper 4-coloring.
- a $(2, 2, 2)$-coloring is a 2-improper 3-coloring.

Known results

Appel and Haken, 1977

- Every planar graph is $(0, 0, 0, 0)$-colorable.

Cowen, Cowen, and Woodall, 1986

- Every planar graph is 2-improperly 3-colorable, i.e. $(2, 2, 2)$-colorable.

Xu, 2009

Every planar graph with neither adjacent triangles nor 5-cycles is $(1, 1, 1)$-colorable.

Known results-Choosability

Definition

A graph G is d-improper m-choosable, or simply $(m, d)^*$-choosable, if for every list assignment L, where $|L(v)| \geq m$ for every $v \in V(G)$, there exists an L-colouring of G such that each vertex of G has at most d neighbours coloured with the same colour as itself.

Eaton and Hull (1999), Škrekovski (1999)

- Every planar graph is 2-improper 3-choosable: $(3, 2)^*$-choosable.

If a graph G is 2-improper 3-choosable then it is $(2, 2, 2)$-colorable.

Škrekovski proved that for every k, there are planar graphs which are not k-improper 2-colorable.

Known results

Cushing and Kierstead (2009)

Every planar graph is 1-improper 4-choosable ($(4,1)^*$-choosable).

Dong and Xu 2009

Let G be a planar graph without any cycles of length in $\{4,8\}$, then G is $(3,1)^*$-choosable.

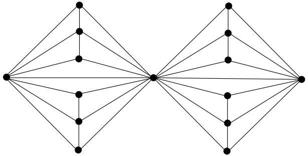
Question (Xu and Zhang, 2007)

Is-it true that every planar graph without adjacent triangle is $(3,1)^*$-choosable.

Every planar graph without adjacent triangle is $(1,1,1)$-colorable?

$(3,1)^*$-choosability

There exist planar graphs containing 4-cycles that are not $(3,1)^*$-choosable (Crown, Crown and Woodall, 1986).



$(3,1)^*$-choosability

Theorem (Chen and R. 2012)

Every planar graph without 4-cycles adjacent to 3- and 4-cycles is $(3,1)^*$-choosable.

Corollary

Every planar graph without 4-cycles is $(3,1)^*$-choosable.

Maximum Average Degree

Definition-Maximum average degree

$$\text{Mad}(G) = max\left\{\frac{2 \cdot |E(H)|}{|V(H)|}, H \subseteq G\right\}.$$

Maximum Average Degree

Definition-Maximum average degree

$$\text{Mad}(G) = max\left\{\frac{2 \cdot |E(H)|}{|V(H)|}, H \subseteq G\right\}.$$

In 1995, Jensen and Toft showed that there is a polynomial algorithm to comput $Mad(G)$ for a given graph G.

T. R. Jensen and B. Toft, Choosability versus chromaticity, Geombinatorics 5(1995), 45-64.

if G is a planar graph with girth g, then $\text{Mad}(G) < \frac{2g}{g-2}$.

(the girth of a graph G is the length of a shortest cycle of G.)

Known results

Havet and Sereni, 2006

- For every $k \geq 0$, every graph G with $Mad(G) < \frac{4k+4}{k+2}$ is k-improperly 2-colorable (in fact k-improperly 2-choosable), i.e. (k,k)-colorable

- $k = 1$ $Mad(G) < \frac{8}{3}$: $(1,1)$-colorable (planar, $g = 8$).
- $k = 2$ $Mad(G) < 3$: $(2,2)$-colorable (planar, $g = 6$).

A more general result:

Theorem (Havet and Sereni)

For every $l \geq 2$ and every $k \geq 0$, all graphs of maximum average degree less than $\frac{l(l+2k)}{l+k}$ are k-improper l-choosable.

it implies $(k, \cdots, k)$-colorable.

$(d_1, d_2, \cdots, d_k)$-coloring

$(d_1, d_2, \cdots, d_k)$-coloring

A graph G is $(d_1, d_2, \cdots, d_k)$-colorable if and only if:

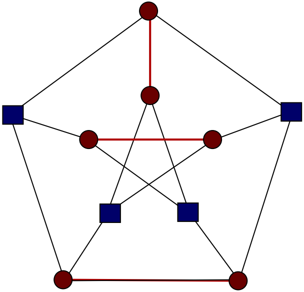
- it exists a partition of V: $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that $\forall i \in [1, k]$, $\Delta(G[V_i]) \leq d_i$

$(d_1, d_2, \cdots, d_k)$-coloring

$(d_1, d_2, \cdots, d_k)$-coloring

A graph G is $(d_1, d_2, \cdots, d_k)$-colorable if and only if:

- it exists a partition of V: $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that $\forall i \in [1, k]$, $\Delta(G[V_i]) \leq d_i$



$(1, 0)$-coloring

$(1, 0)$-colorable

Theorem (Glebov and Zambalaeva, 2007)

Every planar graph is $(1, 0)$*-colorable if* $g(G) \geq 16$.

Theorem (Borodin and Ivanova, 2009)

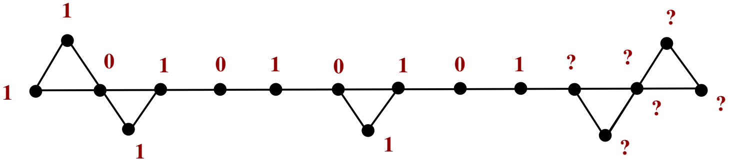
Every graph is $(1, 0)$*-colorable if* $Mad(G) < \frac{7}{3}$

This implies: A planar graph is $(1, 0)$-colorable if $g(G) \geq 14$

Improvement

Theorem (Borodin and Kostochka, 2010)

Every graph G with $Mad(G) \leq \frac{12}{5}$ is $(1,0)$-colorable and the restriction on $Mad(G)$ is sharp.

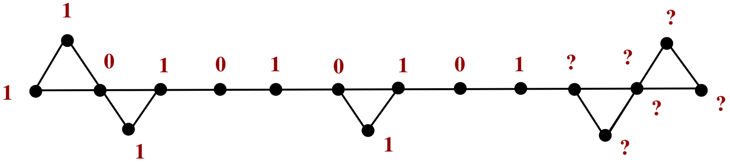


G_3

Improvement

Theorem (Borodin and Kostochka, 2010)

Every graph G with $Mad(G) \leq \frac{12}{5}$ is $(1,0)$-colorable and the restriction on $Mad(G)$ is sharp.



G_3

$$Mad(G_p) = \frac{2|E(G_p)|}{|V(G_p)|} = \frac{12p+6}{5p+2} = \frac{12}{5} + \frac{6}{5(5p+2)}$$

Corollary

A planar graph is $(1,0)$-colorable if $g(G) \geq 12$

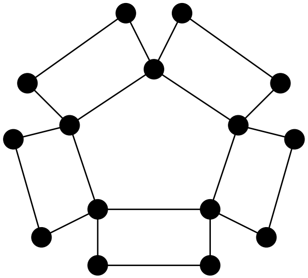
$(k, 0)$-coloring

$(k, 0)$-coloring

- Bipartition V_1,V_2 of $V(G)$
- $\Delta(G[V_1]) \leq k$ and $G[V_2]$ is a stable set.

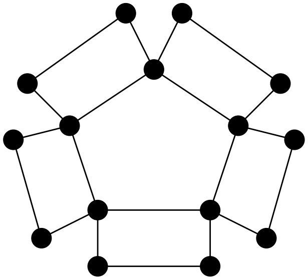
Color k: vertices of V_1
Color 0: vertices of V_2.

Outerplanar graphs



Smallest value of k such that G admits a $(k, 0)$-coloring?

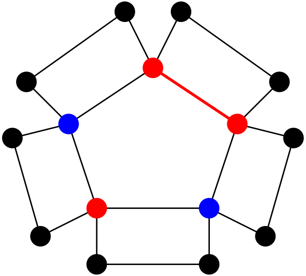
Outerplanar graphs



Smallest value of k such that G admits a $(k, 0)$-coloring?

$k = 1$?

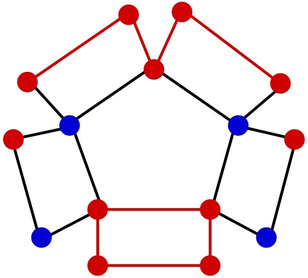
Outerplanar graphs



Smallest value of k such that G admits a $(k, 0)$-coloring?

$k = 1$?

Outerplanar graphs



$(2, 0)$-coloring

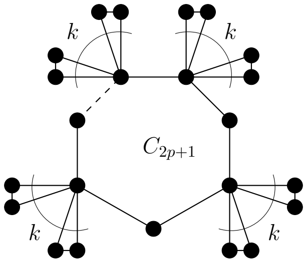
Outerplanar graphs

Claim

- Outerplanar graphs with girth 4 are (2,0)-colorable.
- Outerplanar graphs with girth 5 are (1,0)-colorable.

Outerplanar graphs

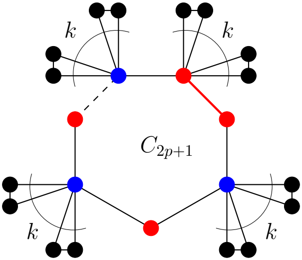
For outerplanar graphs with girth 3, k is unbounded.



Non $(k, 0)$-colorable outerplanar graph with girth 3

Outerplanar graphs

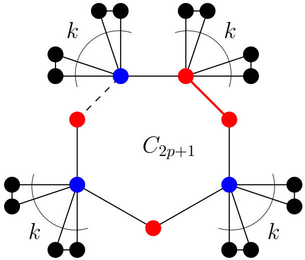
For outerplanar graphs with girth 3, k is unbounded.



Non $(k, 0)$-colorable outerplanar graph with girth 3

Outerplanar graphs

For outerplanar graphs with girth 3, k is unbounded.



Non $(k, 0)$-colorable outerplanar graph with girth 3

$$Mad(G_{p,k}) = \frac{2|E(G_{p,k})|}{|V(G_{p,k})|} = \frac{2((3k+2)(p+1)-1)}{(2k+2)(p+1)-1} \xrightarrow[p\to\infty]{} \frac{3k+2}{k+1}$$

Sparse graphs

Key concepts:

soft components, feading area

Theorem (Borodin, Ivanova, Montassier, Ochem, R., 2009)

Let $k \geq 0$ be a integer. Every graph with maximum average degree smaller than $\frac{3k+4}{k+2}$ is $(k, 0)$-colorable.

- $Mad(G) < \frac{5}{2} \rightarrow (2, 0)$
- $Mad(G) < \frac{13}{5} \rightarrow (3, 0)$

Sparse graphs

Key concepts:

soft components, feading area

Theorem (Borodin, Ivanova, Montassier, Ochem, R., 2009)

Let $k \geq 0$ be a integer. Every graph with maximum average degree smaller than $\frac{3k+4}{k+2}$ is $(k,0)$-colorable.

- $Mad(G) < \frac{5}{2} \rightarrow (2,0)$
- $Mad(G) < \frac{13}{5} \rightarrow (3,0)$

Optimality:

$$\lim_{p \to \infty} Mad(G_{p,k}) = \frac{3k+2}{k+1} < \frac{3k+4}{k+2} + \frac{1}{k+3}\,.$$

Planar graphs

Corollary

Every planar graph G is:

- $(1,0)$-colorable if $g(G) \geq 14$,
- $(2,0)$-colorable if $g(G) \geq 10$,
- $(3,0)$-colorable if $g(G) \geq 9$,
- $(4,0)$-colorable if $g(G) \geq 8$,
- $(8,0)$-colorable if $g(G) \geq 7$.

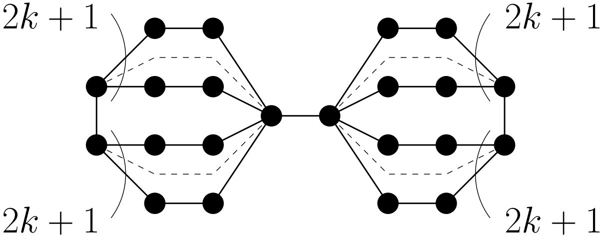
Planar graphs

Corollary

Every planar graph G is:

- $(1, 0)$-colorable if $g(G) \geq 14$,
- $(2, 0)$-colorable if $g(G) \geq 10$,
- $(3, 0)$-colorable if $g(G) \geq 9$,
- $(4, 0)$-colorable if $g(G) \geq 8$,
- $(8, 0)$-colorable if $g(G) \geq 7$.

For planar graphs with girth 6, k is unbounded.



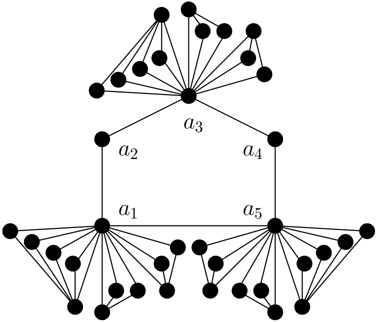
Non $(k,0)$-colorable planar graph with girth 6

$(k, 1)$-coloring

Theorem (Borodin, Ivanova, Montassier, R., 2010)

Let $k \geq 2$ be a integer. Every graph with maximum average degree smaller than $\frac{10k+22}{3k+9}$ is $(k, 1)$-colorable.

$(k, 1)$-coloring



An example of $G_{n,k}$ with $n = 3$ and $k = 3$.

Non $(k, 1)$-colorable graph.

$$Mad(G_{n,k}) = \frac{2|E(G_{n,k})|}{|V(G_{n,k})|} = \frac{2(2n - 1 + 5(k-1)n + n(2k+3))}{2n - 1 + 3(k-1)n + n(k+2)} = \frac{2(7nk - 1)}{n(4k+1) - 1}$$

$$\lim_{n \to \infty} Mad(G_{n,k}) = \frac{14k}{4k+1}$$

(k, j)-coloring: the best result

Let $F(j, k)$ denote the supremum of x such that every graph G with $Mad(G) \leq x$ is (k, j)-colorable.

Theorem (Borodin and Kostochka, 2011)

Let $j \geq 0$ *and* $k \geq 2j + 2$. *Then* $F(j, k) = 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$.

- If Mad(G) $\leq \frac{3k+2}{k+1}$ $(k \geq 2)$ then G is $(k, 0)$-colorable.
- If Mad$(G) \leq \frac{10k+8}{3k+3}$ $(k \geq 4)$ then G is $(k, 1)$-colorable.

$(k,0)$-coloring of planar graphs

Corollary

Every planar graph G is:

- $(1,0)$-colorable if $g(G) \geq 12$ (14),
- $(2,0)$-colorable if $g(G) \geq 8$ (10),
- $(4,0)$-colorable if $g(G) \geq 7$ (8).

$(1,1)$-coloring

Theorem (Borodin, Kostochka and Yancey, 2011)

Every graph G with $Mad(G) \leq 14/5$ is $(1,1)$-colorable and the restriction $Mad(G)$ is sharp.

Summarizing table

For planar graphs:

$g(G)$	$(k,0)$	$(k,1)$	$(k,2)$	$(k,3)$	$(k,4)$
3,4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$	?	$(6,2)$	?	$(4,4)$
6	$\times$	$(4,1)$	$(2,2)$		
7	$(4,0)$	$(1,1)$			
8	$(2,0)$				
12	$(1,0)$				

Question (Montassier, Ochem 2012)

- *Does there exist k such that every planar graph with girth 5 is $(k,1)$-colorable?*
- *Find the smallest value k such that every planar graph with girth 5 is $(k,3)$-colorable? ($k \leq 5$).*

Steinberg Conjecture

Conjecture (Steinberg, 1976)

Every planar graph without 4 and 5-cycles is 3-colorable ((0,0,0)-colorable)

Erdős' relaxation '91: Determine the smallest value of k, if it exists, such that every planar graph without cycles of length from 4 to k is 3-colorable.

- $k \leq 11$ Abbott and Zhou ('91)
- $k \leq 10$ Borodin ('96)
- $k \leq 9$ Borodin ('96) and Sanders and Zhao ('95)
- $k \leq 8$ Salavatipour (2002)
- $k \leq 7$ Borodin et al. (2005)

Relaxation of Steinberg's Conjecture

Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5.
Can we prove that every graph in $\mathcal{F}$ is:

- $(1, 0, 0)$-colorable?

Relaxation of Steinberg's Conjecture

Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5.
Can we prove that every graph in $\mathcal{F}$ is:

- $(1, 0, 0)$-colorable?
- $(1, 1, 0)$-colorable?

Relaxation of Steinberg's Conjecture

Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5.
Can we prove that every graph in $\mathcal{F}$ is:

- $(1, 0, 0)$-colorable?
- $(1, 1, 0)$-colorable?
- $(1, 1, 1)$-colorable (Xu, 2009- Lih, Song, Wang, Zhang, 2000)

Relaxation of Steinberg's Conjecture

Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5.
Can we prove that every graph in $\mathcal{F}$ is:

- $(1, 0, 0)$-colorable?
- $(1, 1, 0)$-colorable?
- $(1, 1, 1)$-colorable (Xu, 2009- Lih, Song, Wang, Zhang, 2000)
- $(2, 0, 0)$-colorable?

Every graph in $\mathcal{F}$ is $(1, 1, 1)$-colorable

Lih, Song, Wang, Zhang, 2000.

By Euler's Formula :

$$|V| - |E| + |F| = 2$$

and

$$\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} r(f)$$

,
and

$$\omega(v) = 2d(v) - 6 \text{ and } \omega(f) = r(f) - 6.$$

we have:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = -12 < 0.$$

If $r(f) = 3$ then $\omega(f) = -3$

Every graph in $\mathcal{F}$ is $(1,1,1)$-colorable

Let G be a minimum counterexample.

1. $\delta(G) \geq 3$
2. no two 3-vertices adjacent
3. no 3-vertex adjacent to two adjacent 4-vertices

Every graph in $\mathcal{F}$ is $(1,1,1)$-colorable

Discharging rules:

Rule 1 v s.t. $d(v) = 4$ gives 1 to each incidente 3-face.

Rule 2 a 5^+-vertex gives 2 to each incidente 3-face.

Remember : a 3-face needs 3.

Every graph in $\mathcal{F}$ is $(1,1,1)$-colorable

Final balance

We have to feed the 3-face.

- $(3,4,5^+)$-face gets 2+1=3
- $(3,5^+,5^+)$-face gets 2+2=4
- $(4^+,4^+,4^+)$-face gets at least 1+1+1=3

We have after discharging $\omega^*(f) \geq 0$ for any face of G

Every graph in $\mathcal{F}$ is $(1,1,1)$-colorable

Final balance

Le v be a vertex of G,

- if v is a 3-vertex, v gives nothing then in this case $\omega^*(v) = \omega(v) = 0$
- if v is a 4-vertex, v can be incident to at most 2 3-faces. Hence $\omega^*(v) \geq 2 - 2 = 0$
- if v is a 5^+-vertex, it can be incident to at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces. Hence: $\omega^*(v) \geq 2d(v) - 6 - 2\lfloor \frac{d(v)}{2} \rfloor \geq 0$

$$0 \leq \sum_{v \in V} \omega^*(v) + \sum_{f \in F} \omega^*(f) = \sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = -12 < 0.$$

A CONTRADICTION.

Every graph in $\mathcal{F}$ is $(1,1,1)$-colorable

Final balance

Le v be a vertex of G,

- if v is a 3-vertex, v gives nothing then in this case $\omega^*(v) = \omega(v) = 0$
- if v is a 4-vertex, v can be incident to at most 2 3-faces. Hence $\omega^*(v) \geq 2 - 2 = 0$
- if v is a 5^+-vertex, it can be incident to at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces. Hence: $\omega^*(v) \geq 2d(v) - 6 - 2\lfloor \frac{d(v)}{2} \rfloor \geq 0$

$$0 \leq \sum_{v \in V} \omega^*(v) + \sum_{f \in F} \omega^*(f) = \sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = -12 < 0.$$

A CONTRADICTION.

Remark

We proved that any $G \in \mathcal{F}$ is $(3,1)^*$-choosable.

Relaxation of Steinberg Conjecture

Theorem (Chang, Havet, Montassier, R. 2011)

Every graph of $\mathcal{F}$ *is* $(4,0,0)$*-colorable.*
Every graph of $\mathcal{F}$ *is* $(2,1,0)$*-colorable.*

Relaxation of Steinberg Conjecture

Theorem (Chang, Havet, Montassier, R. 2011)

Every graph of $\mathcal{F}$ *is* $(4,0,0)$*-colorable.*
Every graph of $\mathcal{F}$ *is* $(2,1,0)$*-colorable.*

Theorem (Hill, Yu 2012. Wang, Xu 2012)

Every graph of $\mathcal{F}$ *is* $(3,0,0)$*-colorable.*
Every graph of $\mathcal{F}$ *is* $(1,1,0)$*-colorable.*

Corollary

Every graph of $\mathcal{F}$ *is* (i,j,k)*-colorable, with* i,j,k *being integers satisfying* $i+j+k=3$.

(i, j, k)-coloring

Theorem (Chen, Montassier, Schiermeyer, R. 2013)

Every graph with $Mad(G) < 7/2$ *is* (i, j, k)*-colorable, with* i, j, k *being integers satisfying* $i + j + k = 3$*.*

(i, j, k)-coloring

Theorem (Chen, Montassier, Schiermeyer, R. 2013)

Every graph with $Mad(G) < 29/8$ *is* $(2, 1, 0)$*-colorable.*

(i, j, k)-coloring

$M(i, j, k) = \inf\{m : m \in \mathbb{R},$
there exists a non-(i, j, k)-colorable graph G with $\mathrm{mad}(G) = m\}$.

$$\begin{aligned} \frac{7}{2} &\leq M(3,0,0) &&\leq \frac{23}{6} + \epsilon \\ \frac{29}{8} &\leq M(2,1,0) &&\leq \frac{22}{5} + \epsilon \\ \frac{15}{4} &\leq M(1,1,1) &&\leq \frac{9}{2} + \epsilon \end{aligned}$$

For any real $\epsilon > 0$.

(i, j, k)-coloring-A non-$(3, 0, 0)$-colorable graph

Claim

For any real $\epsilon > 0$, there exists a non-$(3, 0, 0)$-colorable graph with $\text{mad}(G) < \frac{23}{6} + \epsilon$.

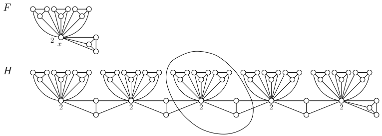


Figure : Graphs F and H.

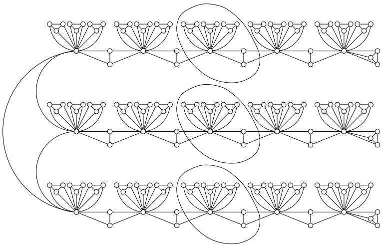


Figure : Non-$(3, 0, 0)$-colorable graph G.

(i, j, k)-coloring-A non-$(2, 1, 0)$-colorable graph

Claim

For any real $\epsilon > 0$, there exists a non-$(2, 1, 0)$-colorable graph with $\text{mad}(G) < \frac{22}{5} + \epsilon$.

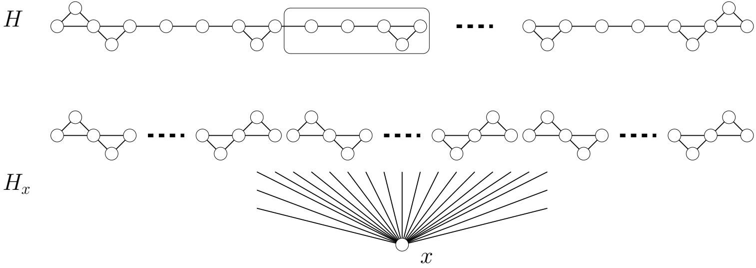


Figure : The graphs H and H_x.

(i, j, k)-coloring-A non-$(1, 1, 1)$-colorable graph

Claim

For any real $\epsilon > 0$, there exists a non-$(1, 1, 1)$-colorable graph with $\text{mad}(G) < \frac{9}{2} + \epsilon$.

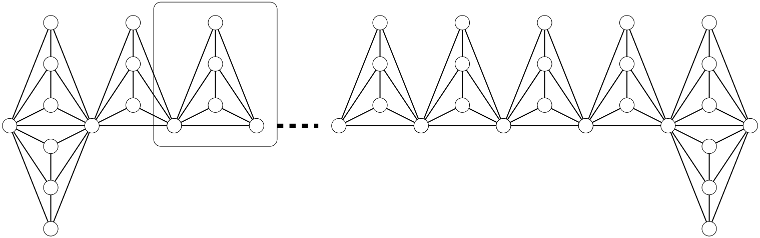
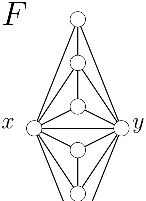
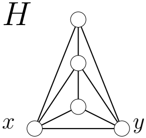


Figure : Non-$(1, 1, 1)$-colorable graph G.





Thank you for your attention!