Planar 4-critical graphs

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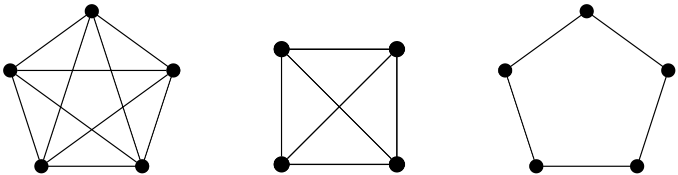
Definitions (k-critical graphs)

graph $G = (V, E)$

G is a *k-critical graph* if G is not $(k-1)$-colorable
but every $H \subset G$ is $(k-1)$-colorable.



Inspiration

Theorem (Grötzsch; 59)
Every planar triangle-free graph is 3-colorable.

Recently reproved by Kostochka and Yancey using

Theorem (Kostochka and Yancey; 12)
If G is 4-critical graph, then

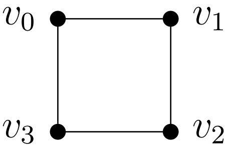
$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

Every planar triangle-free graph is 3-colorable.

Let G be a minimal counterexample - i.e. G is 4-critical

$|E(G)| = e, |V(G)| = v, |F(G)| = f$.

CASE1 G contains a 4-face



CASE2 G contains no 4-faces

Every planar triangle-free graph is 3-colorable.

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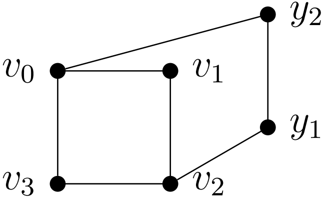

CASE2 G contains no 4-faces

Every planar triangle-free graph is 3-colorable.

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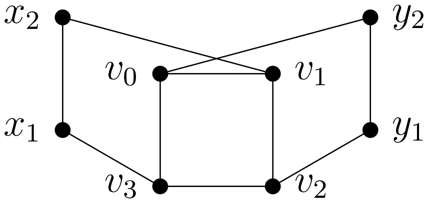


CASE2 G contains no 4-faces

Every planar triangle-free graph is 3-colorable.

Let G be a minimal counterexample - i.e. G is 4-critical

$|E(G)| = e, |V(G)| = v, |F(G)| = f$.

CASE1 G contains a 4-face

CASE2 G contains no 4-faces

- $v + f = e + 2$ by Euler's formula
- $2e \geq 5f$ since face is at least 5-face
- $5v + 5f = 5e + 10$
- $3e \leq 5v - 10$ (our case)
- $3e \geq 5v - 2$ (every 4-critical graph)

Generalizations?

Theorem (Grötzsch; 59)

Every planar triangle-free graph is 3-colorable.

Can be strengthened?

Yes! - recall that CASE2

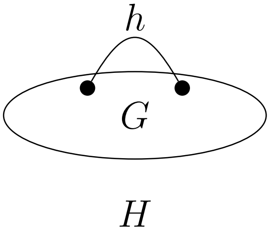
- $3e \leq 5v - 10$ (3-,4- faces free)
- $3e \geq 5v - 2$ (every 4-critical graph)

has some gap.

Adding a bit

Theorem (Aksenov 77; Jensen and Thomassen 00)

Let G be a triangle-free planar graph and H be a graph such that $G = H - h$ for some edge h of H. Then H is 3-colorable.



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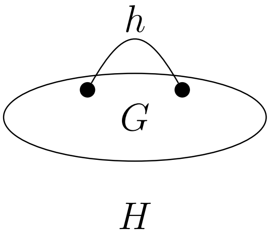
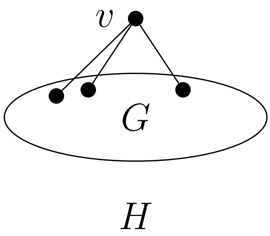
Theorem (Jensen and Thomassen 00)

Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 3. Then H is 3-colorable.





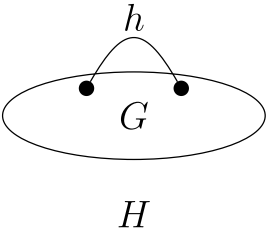
Adding a bit

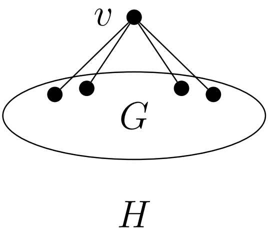
Theorem (Aksenov 77; Jensen and Thomassen 00)

Let G be a triangle-free planar graph and H be a graph such that $G = H - h$ for some edge h of H. Then H is 3-colorable.

Theorem

Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 4. Then H is 3-colorable.

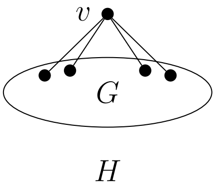




For proof

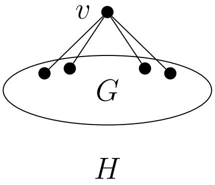
Theorem

Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 4. Then H is 3-colorable.



Proof

G plane, triangle-free, $G = H - v$,
H 4-critical



CASE1: No 4-faces in G

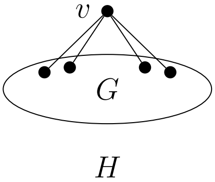
$V(H) = v, E(H) = e, V(G) = v - 1, E(G) = e - 4, F(G) = f$

- $5f \leq 2(e-4)$ since G has no 4-faces
- $(n-1) + f - (e-4) = 2$ by Euler's formula
- $5n - 3e - 8 \geq -5$ together, so $3e \leq 5n - 3$
- but $e \geq \frac{5n-2}{3}$ by 4-criticality of H

CASE2: 4-face $(x_1, x_2, x_3, x_4) \in G$

Proof

G plane, triangle-free, $G = H - v$,
H 4-critical



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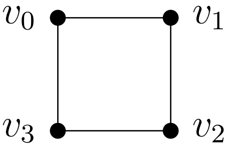
CASE2: 4-face $(x_1, x_2, x_3, x_4) \in G$



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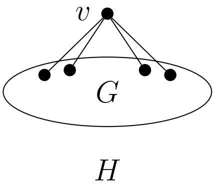

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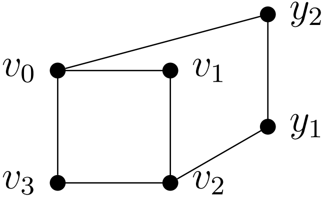

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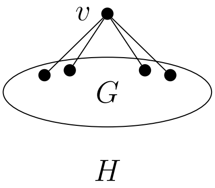

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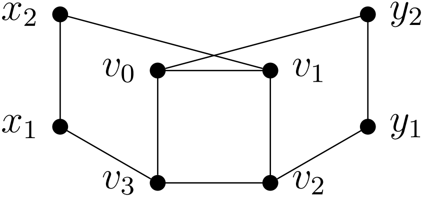
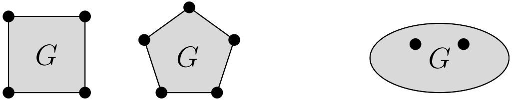


Precoloring

Theorem (Grötzsch 59)

Let G be a triangle-free planar graph and F be a face of G of length at most 5. Then each 3-coloring of F can be extended to a 3-coloring of G.



Theorem (Aksenov et al. 02)

Let G be a triangle-free planar graph. Then each coloring of two non-adjacent vertices can be extended to a 3-coloring of G.

For proof

Theorem (Grötzsch 59)

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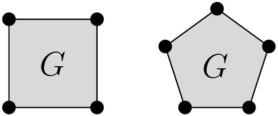
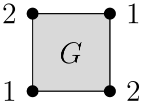


Proof

If G is triangle-free planar, F is a precolored 4-face or 5-face, then precoloring of F extends.

CASE1: F is a 4-face

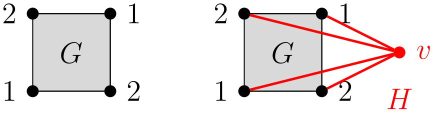


CASE2: F is a 5-face

Proof

If G is triangle-free planar, F is a precolored 4-face or 5-face, then precoloring of F extends.

CASE1: F is a 4-face H is 3-colorable



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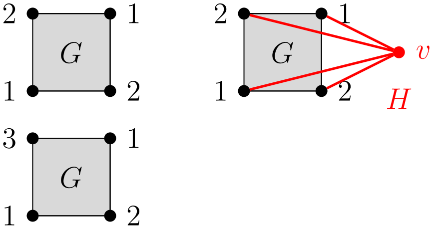

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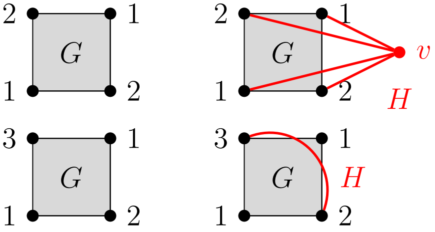

CASE2: F is a 5-face

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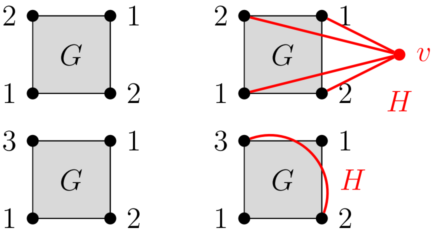
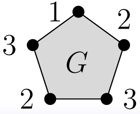


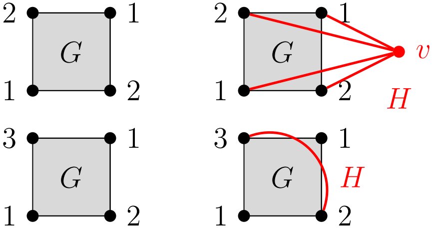
CASE2: F is a 5-face



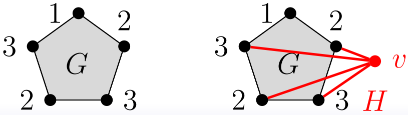
Proof

If *G* is triangle-free planar, *F* is a precolored 4-face or 5-face, then precoloring of *F* extends.

CASE1: *F* is a 4-face *H* is 3-colorable



CASE2: *F* is a 5-face

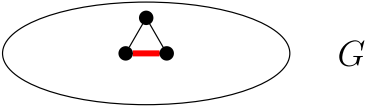


Some triangles?

Theorem (Grötzsch; 59)

Every planar triangle-free graph is 3-colorable.

We already showed one triangle!



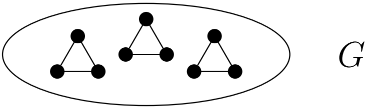
Some triangles?

Theorem (Grötzsch; 59)

Every planar triangle-free graph is 3-colorable.

Theorem (Grünbaum 63; Aksenov 74; Borodin 97)

Let G be a planar graph containing at most three triangles. Then G is 3-colorable.



Three triangles - Proof

Theorem (Grünbaum 63; Aksenov 74; Borodin 97)

Let G be a planar graph containing at most three triangles. Then G is 3-colorable.

- G is 4-critical (minimal counterexample)
- 3-cycle is a face
- 4-cycle is a face or has a triangle inside and outside
- 5-cycle is a face or has a triangle inside and outside

CASE1: G has no 4-faces
CASE2: G has a 4-faces with triangle (no identification)
CASE3: G has a 4-face where identification applies

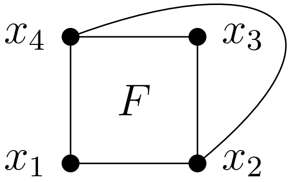
Three triangles - Proof

CASE1: G has no 4-faces

- $v + f = e + 2$ by Euler's formula
- $2e \geq 5f - 6$ since face is at least 5-face and 3 triangles
- $5v + 5f = 5e + 10$
- $3e \leq 5v - 4$ (our case)
- $3e \geq 5v - 2$ (every 4-critical graph)

Three triangles - Proof

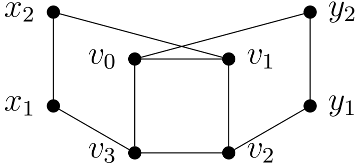
CASE2: G has a 4-face F with a triangle (no identification)



Both x_1, x_2, x_4 and x_2, x_3, x_4 are faces.

Three triangles - Proof

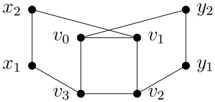
CASE3: *G* has a 4-face where identification applies



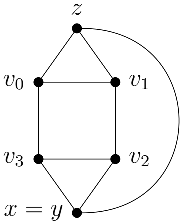
Since *G* is planar, some vertices are the same.

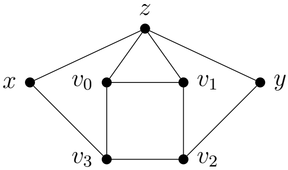
Three triangles - Proof

CASE3: G has a 4-face where identification applies



Since G is planar, some vertices are the same.





Even more triangles?

Let G be a 4-critical planar graph with four triangles and no 4-faces.

- $3e \leq 5v - 2$ (3-,4- faces free)
- $3e \geq 5v - 2$ (every 4-critical graph)

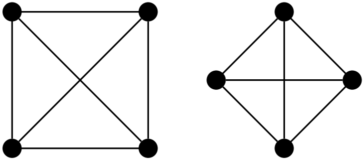
Hence $3e = 5v - 2$.

4 triangles - main tool

Theorem (Kostochka and Yancey; 12)

Let G be a 4-critical graph. Then $e = \frac{5v-2}{3}$ iff G is 4-Ore.

G is 4-Ore if $G = K_4$ or G is an Ore composition of two 4-Ore graphs.



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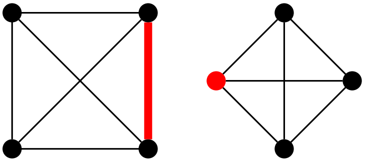

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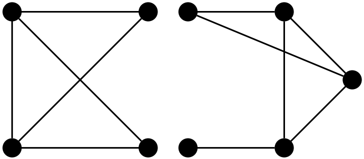

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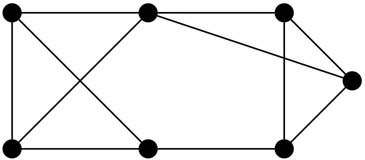

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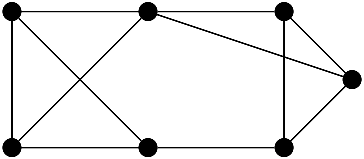
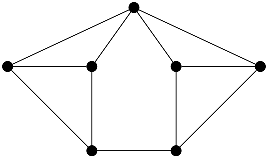




Key properties

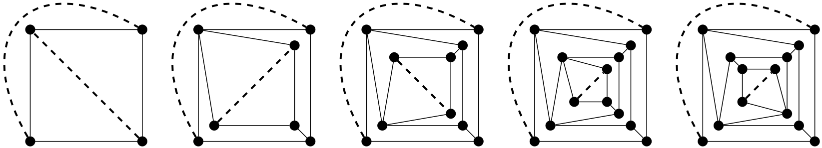
G is $4, 4$-*graph* if it is 4-Ore and has 4 triangles

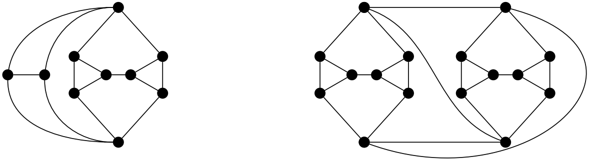
- $4, 4$-graph is K_4 or Ore composition of two $4, 4$-graphs
- The removed edge in Ore composition must be in two triangles
- the split vertex must be in two triangles

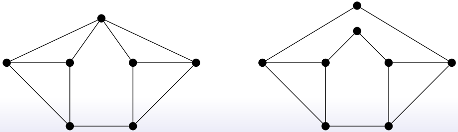
Lemma

Every $4, 4$-graph is planar.

Description (by picture)







Thank you for your attention!