3-coloring planar graphs with four triangles

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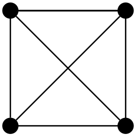
Definitions (4-critical graphs)

graph $G = (V, E)$

coloring is $\varphi : V \to C$ such that $\varphi(u) \neq \varphi(v)$ if $uv \in E$

G is a *k-colorable* if coloring with $|C| = k$ exists

G is a *4-critical graph* if G is not 3-colorable
but every $H \subset G$ is 3-colorable.



Inspiration

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

Recently reproved by Kostochka and Yancey using

Theorem (Kostochka and Yancey '12)

If G is 4-critical graph, then

$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

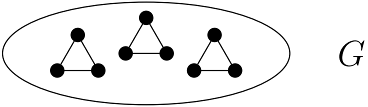
More triangles?

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

Theorem (Grünbaum '63; Aksenov '74; Borodin '97; Borodin et. al. '12+)

Let G be a planar graph containing at most three triangles. Then G is 3-colorable.



Question: What about four triangles?

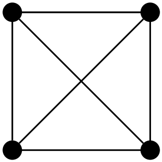
3-coloring planar graphs with four triangles?

Characterizing a 4-critical planar graphs with 4 triangles is a problem of Erdős ’92.

Some partial results announced by Borodin ’97

3-coloring planar graphs with four triangles?

A graph G is not always 3-colorable if it contains 4-triangles: K_4



For K_4:
$3e = 5v - 2$ as $3 \cdot 6 = 5 \cdot 4 - 2$

Kostochka and Yancey gave more properties of 4-critical graphs if $3e = 5v - 2$.

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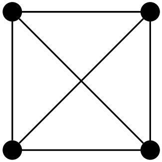

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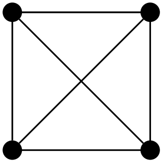

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3-coloring planar graphs with four triangles?

Let G be a 4-critical planar graph with four triangles and no 4-faces.

- $3e \leq 5v - 2$ (four triangles and no 4-faces)
- $3e \geq 5v - 2$ (every 4-critical graph)

Hence $3e = 5v - 2$.

Kostochka and Yancey giving more properties. But only when no 4-faces!

PLAN: Try to characterize 4-critical planar graph with four triangles and no 4-faces. Use it to get all 4-critical planar graphs (even with 4-faces).

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Results

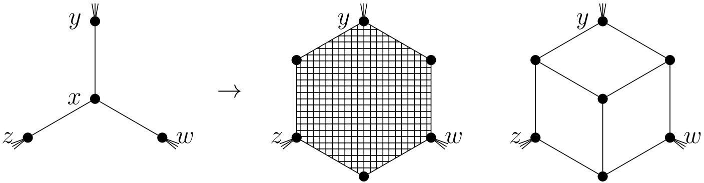
Theorem

4-critical planar graphs without 4-faces are precisely graphs in $\mathcal{C}$.

$\mathcal{C}$ is described later...

Theorem

Every 4-critical planar graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.



Act 1:

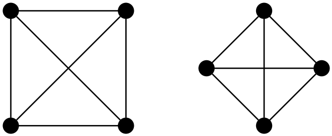
Theorem
4-critical planar graphs without 4-faces are precisely graphs in $\mathcal{C}$.

Main tool:

Theorem (Kostochka and Yancey; 12+)

Let G be a 4-critical graph. Then $3e = 5v - 2$ iff G is 4-Ore.

G is 4-Ore if $G = K_4$ or G is an Ore composition of two 4-Ore graphs.



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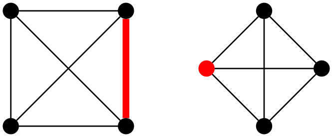

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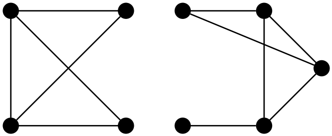

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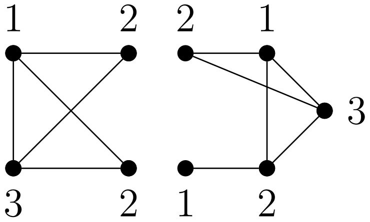

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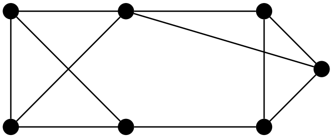

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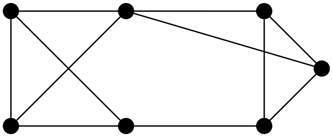
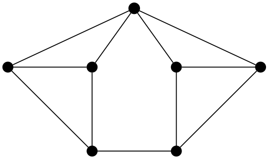


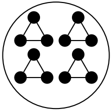


Not 3-colorable.

Key properties

G is $4,4$-*graph* if it is 4-Ore and has 4 triangles

- $4,4$-graph is K_4 or Ore composition of two $4,4$-graphs



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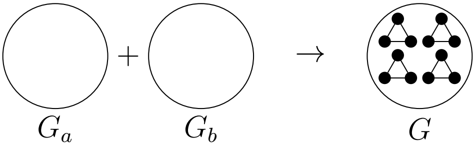

If $G \neq K_4$, then G is an Ore composition of G_a and G_b.

Key properties

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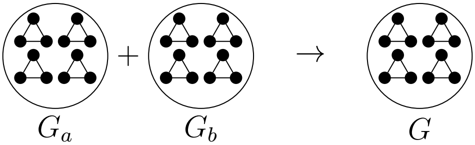

Both G_a and G_b have at least four triangles.

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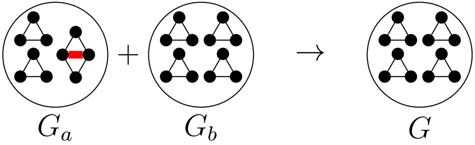

One edge can be in at most two triangles $\Rightarrow$ at least 2 triangles survive in G_a.

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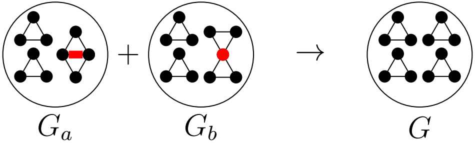

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If $G_b \neq K_4$, then every vertex in at most two triangles $\Rightarrow$ at least 2 triangles survive in G_b.

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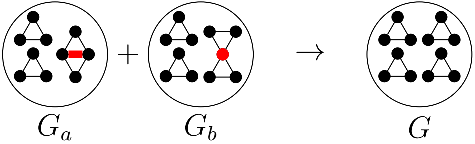
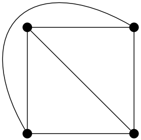
G is $4,4$-*graph* if it is 4-Ore and has 4 triangles

- $4,4$-graph is K_4 or Ore composition of two $4,4$-graphs



One edge can be in at most two triangles $\Rightarrow$ at least 2 triangles survive in G_a.
If $G_b \neq K_4$, then every vertex in at most two triangles $\Rightarrow$ at least 2 triangles survive in G_b.
Conclusion: G_a and G_b are $4,4$-graphs

Description of $4, 4$-graphs (by pictures)



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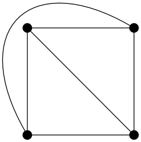

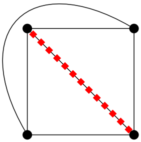

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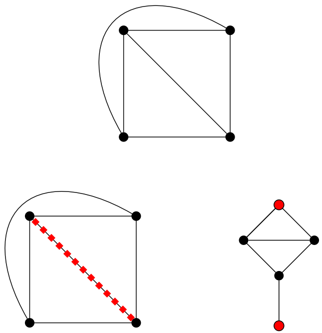

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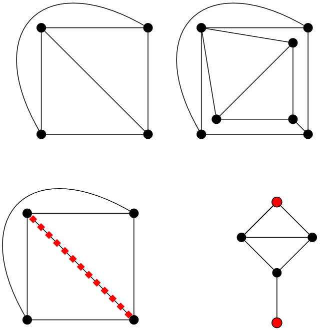

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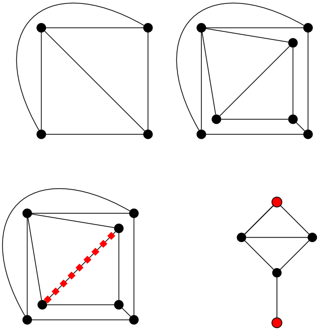

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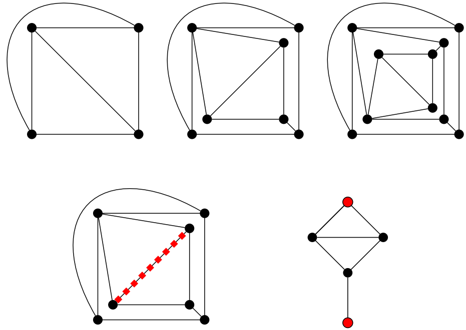

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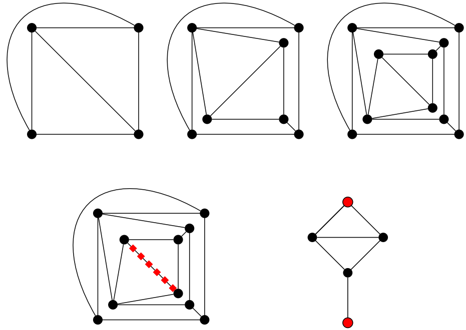

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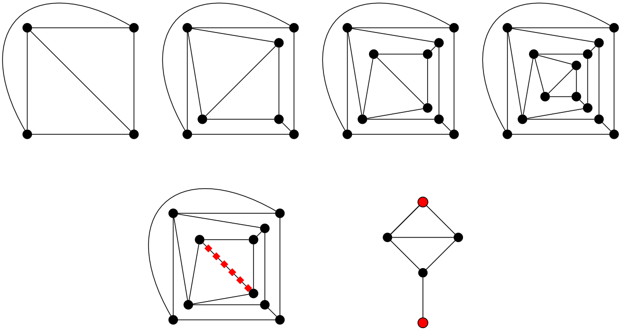

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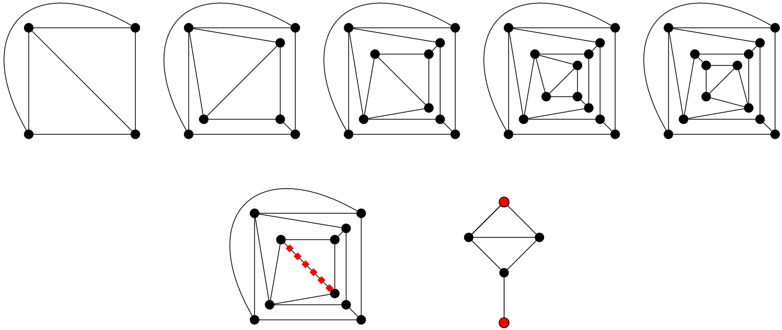

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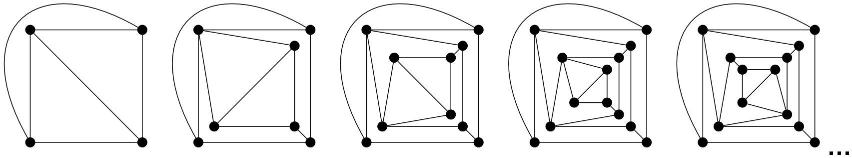

Infinite class - same as Thomas-Walls for the Klein bottle without contractible 3- and 4-cycles.

And now few more...

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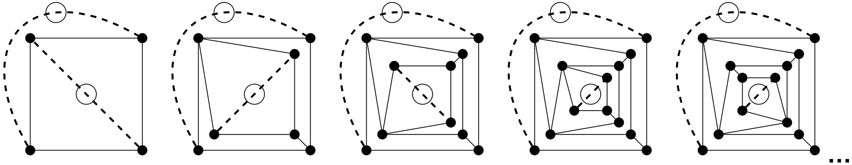

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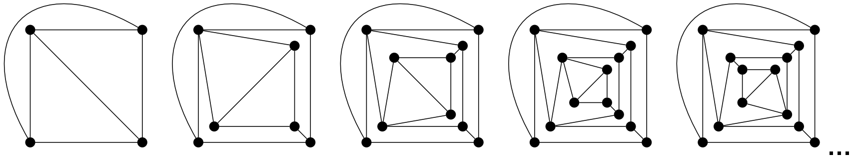

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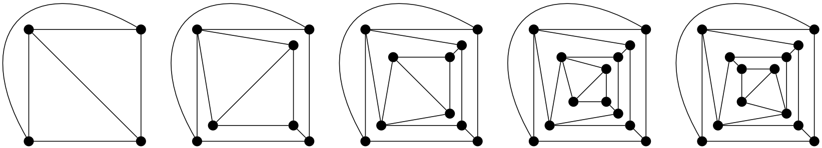

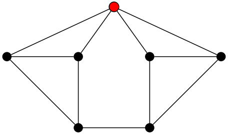

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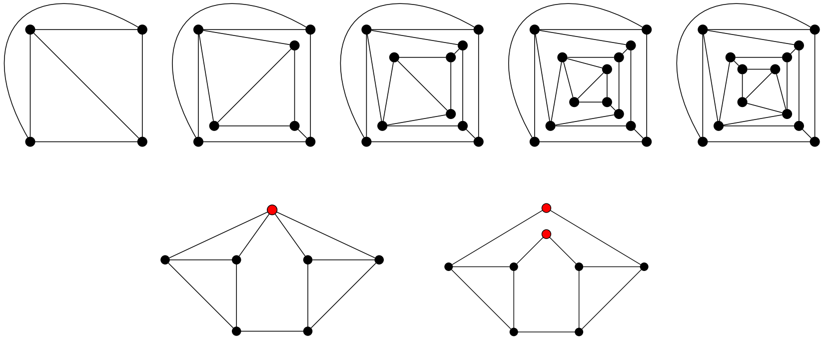

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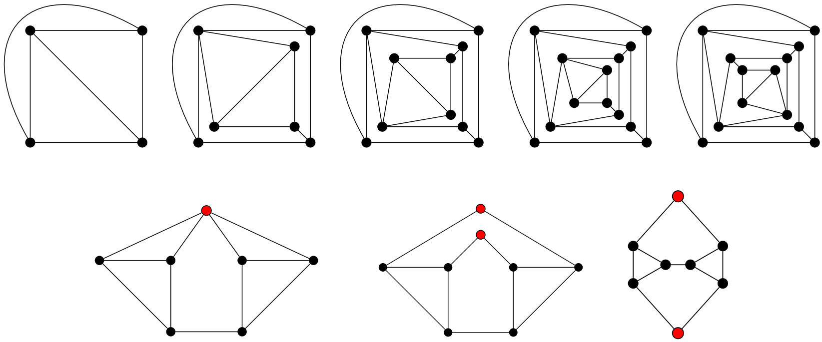
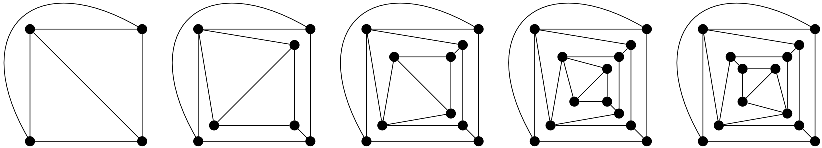
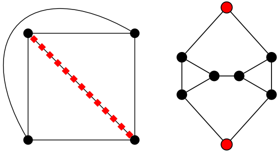


Description of $4, 4$-graphs (by picture)





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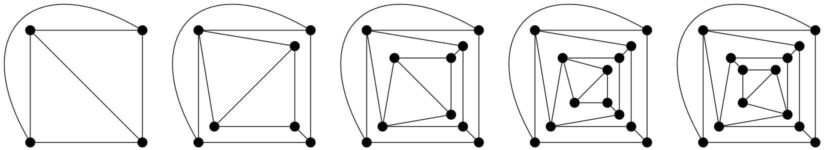

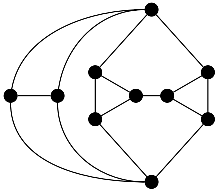

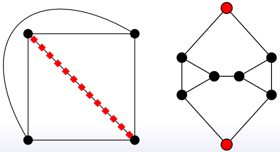

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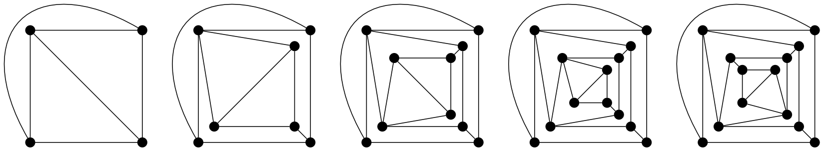

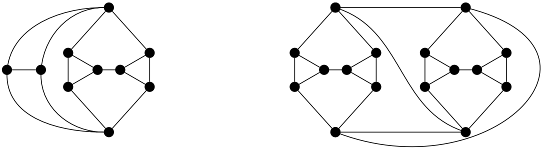

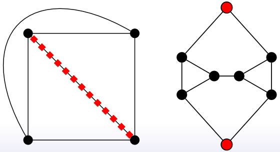

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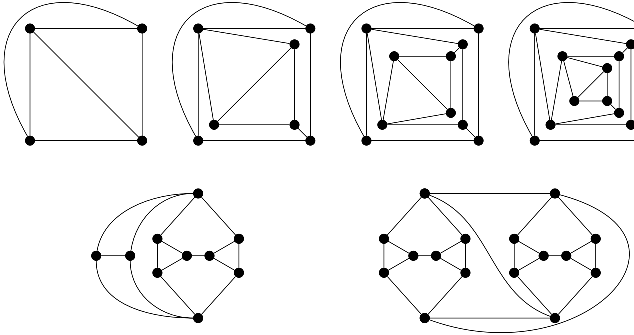
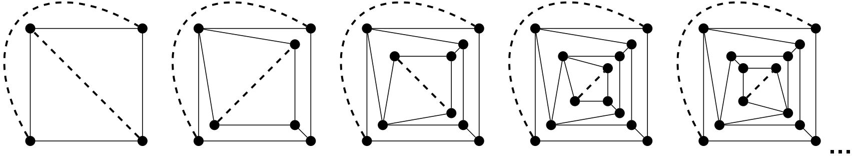


Lemma

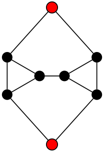
Every $4, 4$-graph is planar.

Description of $\mathcal{C}$

All 4-critical planar graphs with four triangles and no 4-faces can be obtained from the Thomas-Walls sequence



by replacing dashed edges by edges or



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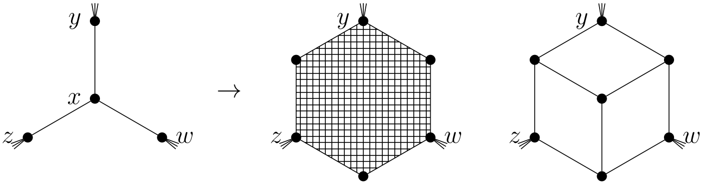
Act 2:

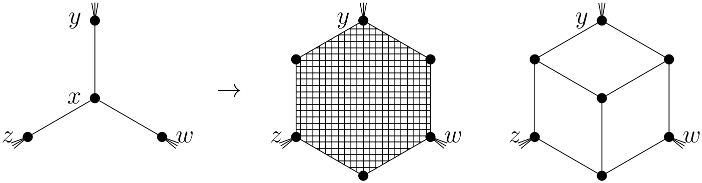
Theorem

Every 4*-critical planar graph can be obtained from* $G \in \mathcal{C}$ *by expanding some vertices of degree 3.*



(Interior of a 6-cycle is a quadrangulation - only 4-faces)

Why is expansion good?



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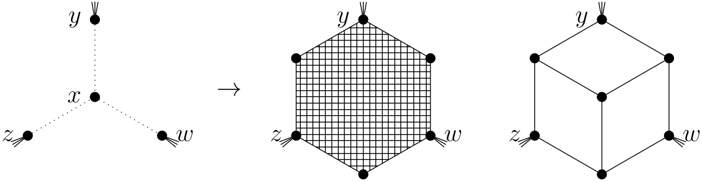

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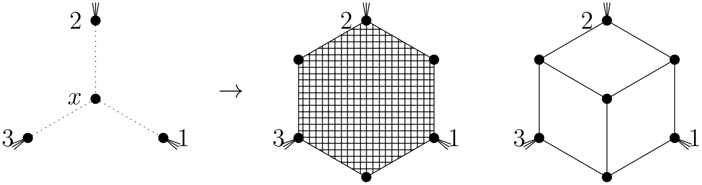
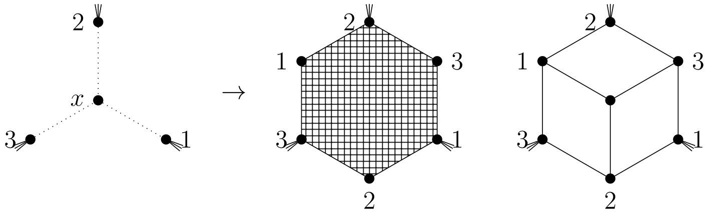


$G - x$ is 3-colorable since G is 4-critical.

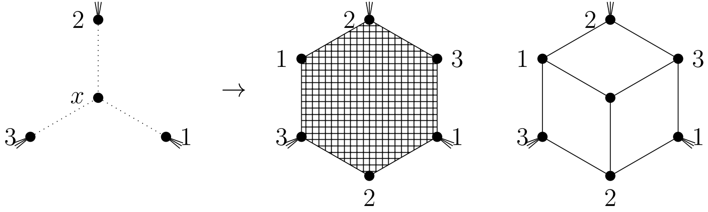
Any 3-coloring of $G - x$ gives different colors to y, z, w.

Why is expansion good?



$G - x$ is 3-colorable since G is 4-critical.
Any 3-coloring of $G - x$ gives different colors to y, z, w.
3-coloring extends to a 3-coloring of 6-cycle uniquely.

Why is expansion good?



Theorem (Gimbel and Thomassen '97)

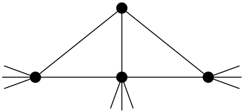
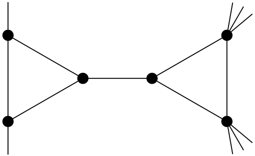
Let G be a planar triangle-free graph with chordless outer 6-cycle C. Let c be a coloring of C by colors 1,2,3. Then c can be extended to a 3-coloring of G if and only if G doesn't contain a 2-connected subgraph H with outer cycle C such that all other facial cycles cycles are 4-cycles and such that opposite vertices of C have the same color.

Theorem

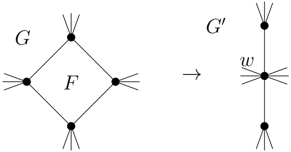
Every 4-critical planar graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.

Let G be a minimal counterexample.

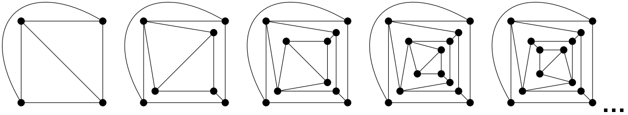
- G has no separating 4-cycle
- G has a 4-face F
- G has does not contain

- Let G' be obtained from G by identifying opposite vertices of F to w (while creating no new triangles)



- Let G'' be a 4-critical subgraph of G'
    - G'' has no 4-faces
    - G'' was described in Act 1.
    - All separating 4-cycles in G'' contain w



There are only few candidates for G''! Not all from Act 1.

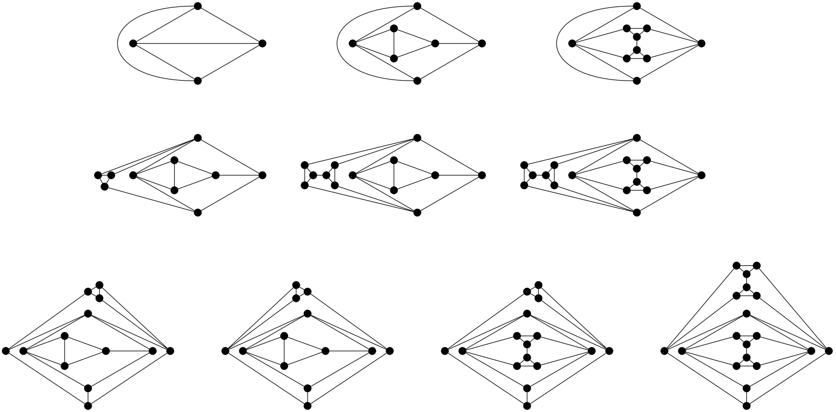
- Let G' be obtained from G by identifying opposite vertices of F to w (while creating no new triangles)

- Let G'' be a 4-critical subgraph of G'
  - G'' has no 4-faces
  - G'' was described in Act 1.
  - All separating 4-cycles in G'' contain w
- Reconstruct G from G'' by guessing w, decontractig w and adding other vertices that were removed.

$$G \xrightarrow{\text{identification}} G' \xrightarrow{\text{critical subgraph}} G''$$

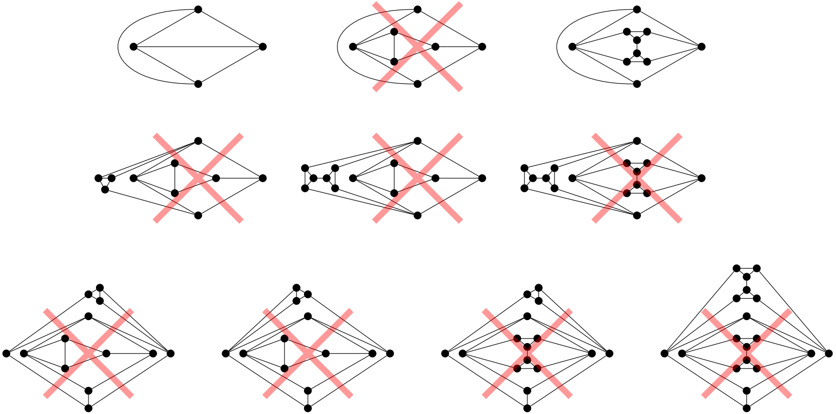
$$G \xleftarrow{\text{adding vertices}} G_1 \xleftarrow{\text{decontraction}} G''$$

Options for G''

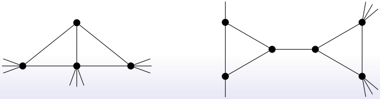


Decontraction changes at most two faces
must destroy all separating 4-cycles

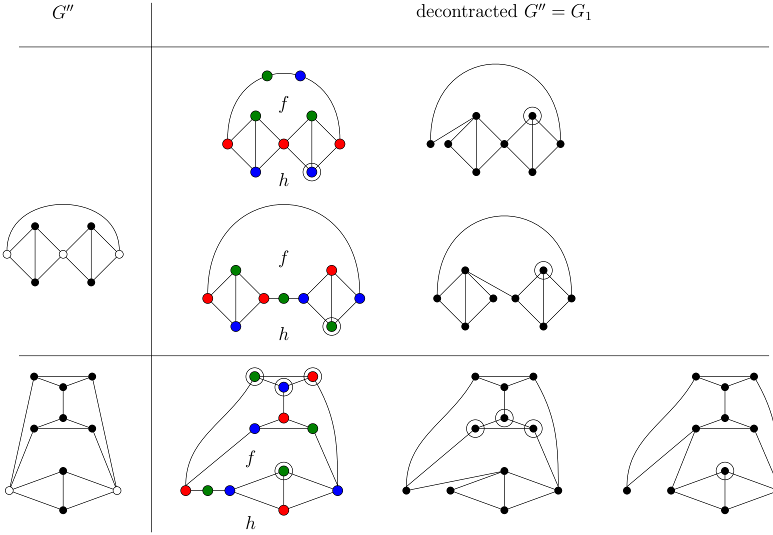
Options for G''



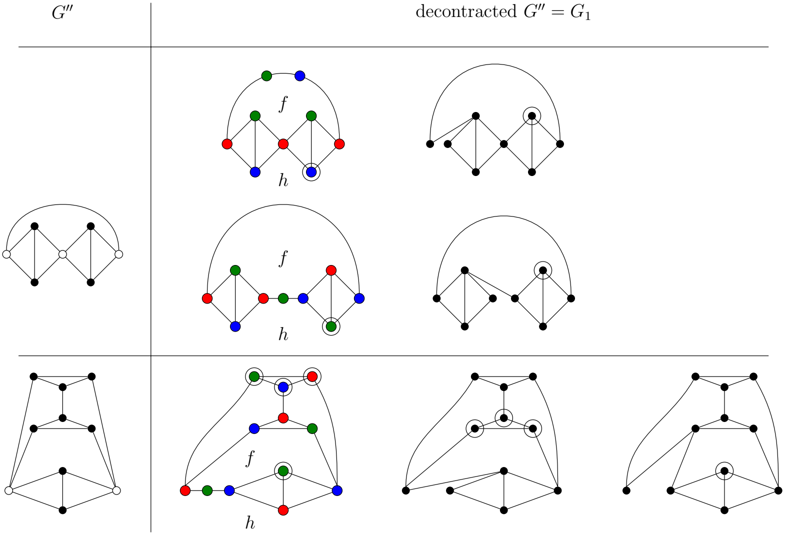
After excluding forbidden configurations in G.



Decontraction of G''



Decontraction of G''

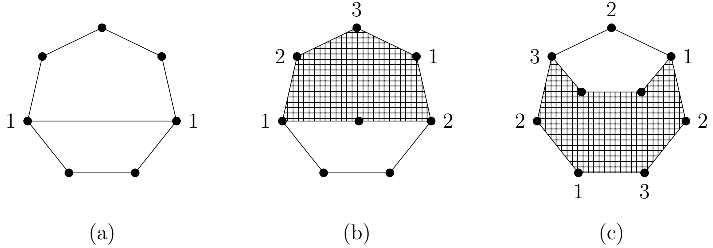


How to fill 7-faces f and h of G_1?

Act 3:

Theorem (Aksenov, Borodin, and Glebov ’04)

Let G be a planar triangle-free graph with outer face bounded by a cycle C of length 7. G is C-critical if and only if G and the precoloring looks like one of



G is *C-critical* if exists a 3-coloring of C not extending to G but extending to any proper subgraph of G containing C.
We show an alternative proof.

Thank you for your attention!