Saturation Number of Ramsey-Minimal Graphs for Matchings

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$\mathcal{F}$-saturated graphs

Definition

Given a family $\mathcal{F}$ of graphs, G is **$\mathcal{F}$-saturated** if:

1. G contains no member of $\mathcal{F}$, and
2. for any pair of nonadjacent vertices u and v in G, $G + uv$ contains some member of $\mathcal{F}$.

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If $\mathcal{F} = \{F\}$, we then say that G is **F-saturated**.

The Turán Problem

Problem (The Turán Problem)

Determine $ex(n, \mathcal{F})$, the maximum number of edges in a graph that contains no member of $\mathcal{F}$ as a subgraph.

$ex(n, \mathcal{F})$ is the **extremal** or **Turán** number of $\mathcal{F}$.

$\text{sat}(n, \mathcal{F})$

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Definition

The *minimum* number of edges in an $\mathcal{F}$-saturated graph is denoted **sat$(\mathbf{n}, \mathcal{F})$**.

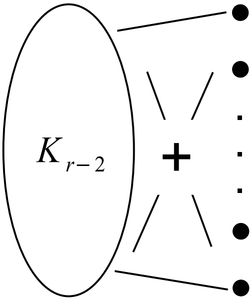
Erdős-Hajnal-Moon

Erdős, Hajnal and Moon determined $sat(n, K_r)$ exactly.

Theorem (E-H-M 1964)

$$sat(n, K_r) = e(K_{r-2} + \overline{K}_{n-r+2}) = \binom{r-2}{2} + (r-2)(n-r+2).$$

Furthermore, $K_r + K_{n-r+2}$ is the unique K_r-saturated graph of minimum size.



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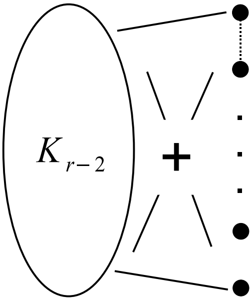

Non-Monotonicity

Interestingly, $\mathrm{sat}(n, \mathcal{F})$ does not share many of the nice properties of $\mathrm{ex}(n, \mathcal{F})$.

$$\mathrm{sat}(n, F) \not\leq \mathrm{sat}(n+1, F)$$

$$\mathrm{sat}(2k-1, P_4) = k+1 > \mathrm{sat}(2k, P_4) = k$$

$$F' \subset F \not\Rightarrow \mathrm{sat}(n, F') \leq \mathrm{sat}(n, F)$$

$$\mathrm{sat}(n, K_{1,m}) > \mathrm{sat}(n, K_{1,m} + e)$$

$$\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow \mathrm{sat}(n, \mathcal{F}_1) \geq \mathrm{sat}(n, \mathcal{F}_2)$$

$$\mathrm{sat}(n, \{K_{1,m} + e\}) < \mathrm{sat}(n, \{K_{1,m}\}) = \mathrm{sat}(n, \{K_{1,m}, K_{1,m} + e\})$$

(Some) Known Results

$sat(n, H)$ has been studied for many classes of graphs.

- $K_{1,t}$ and P_t (Kásonyi and Tuza 1986)
- Matchings (Mader 1973, Kásonyi and Tuza 1986)
- tK_r and $K_r \cup K_s$ (Faudree, Ferrara, Gould and Jacobson 2009)
- Trees (Faudree, Faudree, Gould, Jacobson 2009)
- *A Survey of Minimum Saturated Graphs* (Faudree, Faudree, Schmitt - submitted)

Theorem (Erdős-Stone-Simonovits)

If H is a nontrivial graph, then

$$\mathrm{ex}(n, H) = (1 - \frac{1}{\chi(H) - 1})\frac{n^2}{2} + o(n^2).$$

Specifically, if H is not bipartite,

$$\mathrm{ex}(H, n) = \Theta(n^2).$$

Bounding the *sat* Function

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Theorem (Kásonyi and Tuza 1986)

$$\mathrm{sat}(n, \mathcal{F}) = O(n).$$

Best known lower bound??

Every large enough H-saturated graph G has

$$\delta(G) \geq \delta(H) - 1,$$

so

$$sat(n, H) \geq \frac{\delta(H)-1}{2} n.$$

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Problem

For an arbitrary graph F determine a non-trivial lower bound on $sat(n, F)$.

Ramsey Numbers

Definition

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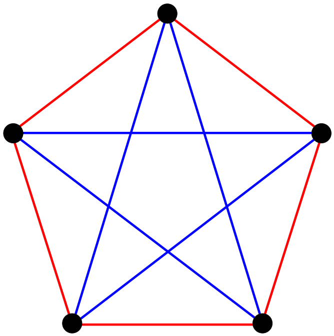
For instance, $r(K_3, K_3) = 6$.



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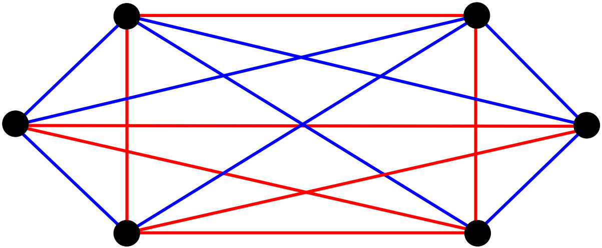

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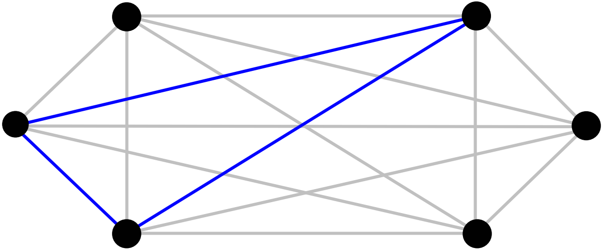

Why Only Complete Graphs?

Definition

Given graphs $G, H_1, \ldots, H_k$, we write

$$G \hookrightarrow (H_1, H_2, \ldots, H_k)$$

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Therefore, $r(H_1, \ldots, H_k)$ is the smallest n such that

$$K_{n-1} \not\hookrightarrow (H_1, \ldots, H_k),$$

and

$$K_n \hookrightarrow (H_1, \ldots, H_k).$$

A Problem of Hanson and Toft

In 1987, Hanson and Toft posed the following problem:

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Determine the minimum number of edges in a graph G of order n such that

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2. *for any $uv \in \overline{G}$, every k-edge-coloring of $G + uv$ contains a monochromatic copy of K_{t_i} in color i for some i.*

$(H_1, \ldots, H_k)$-Ramsey Minimality

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Given graphs $G, H_1, \ldots, H_k$, we say that G is **$(H_1, \ldots, H_k)$-Ramsey-minimal** if $G \hookrightarrow (H_1, \ldots, H_k)$, but $G' \not\hookrightarrow (H_1, \ldots, H_k)$ for any proper subgraph G' of G.

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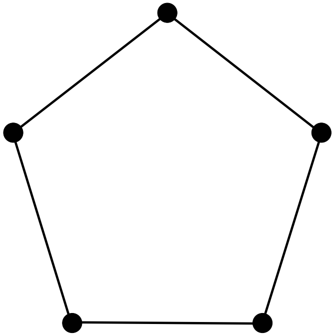
Example: C_5 is (P_3, P_3)-Ramsey-minimal.



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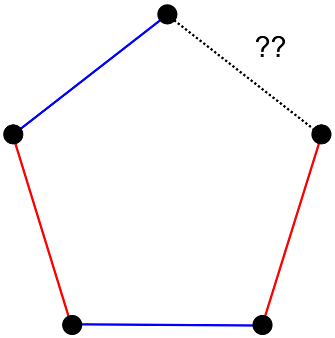

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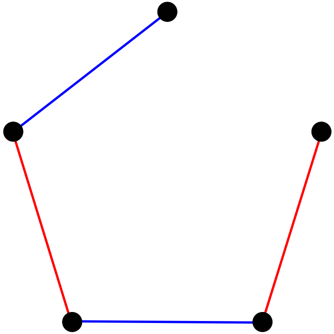

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but for any $uv \in \overline{G}$,

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An Important Observation:

Let $\mathcal{R}_{min}(H_1, \ldots, H_k)$ denote the family of $(H_1, \ldots, H_K)$-Ramsey minimal graphs.

Problem (Hanson and Toft)

Determine $sat(n, \mathcal{R}_{min}(K_{t_1}, \ldots, K_{t_k}))$.

The Hanson-Toft Conjecture

Conjecture (Hanson and Toft 1987)

Let $r = r(K_{t_1}, \ldots, K_{t_k})$. *Then*

$$sat(n, \mathcal{R}_{min}(K_{t_1}, \ldots, K_{t_k})) = sat(n, K_r).$$

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If $t_i \geq 3$ for at most one i, the conjecture follows from Erdős-Hajnal-Moon.

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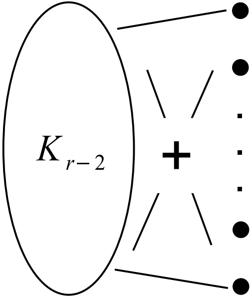

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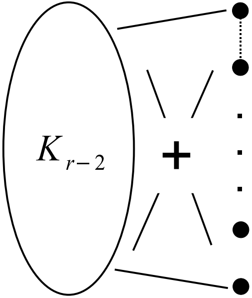

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Theorem (Chen, Ferrara, Gould, Magnant, Schmitt 2011)

For $n \geq 56$,

$$sat(n, R_{min}(K_3, K_3)) = sat(n, K_6) = 4n - 10.$$

Easy Cases

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Problem

$$sat(n, R_{min}(m_1 K_2, m_2 K_2, \cdots, m_k K_2)?$$

Main Result

Theorem

For $n > 3(m_1 + \cdots + m_k - k)$,

$$sat(n, R_{min}(m_1K_2, m_2K_2, \cdots, m_kK_2) = 3(m_1 + m_2 + \cdots + m_k - k)$$

If $m_i \leq 2$ *for all* i*, the extremal graph is union of edge disjoint triangles and isolated vertices. Otherwise, the unique extremal graph is union of vertex disjoint triangles and isolated vertices..*

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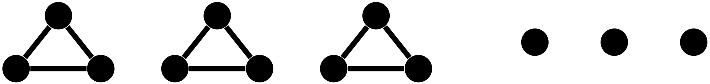

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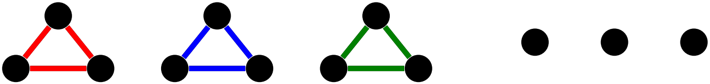

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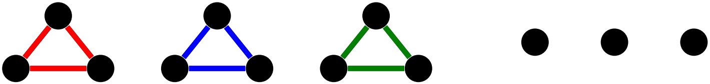

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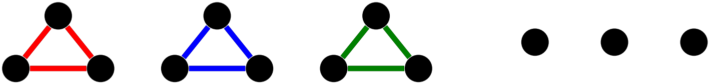

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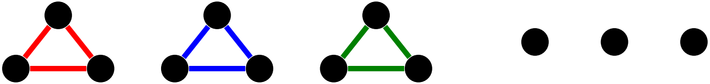

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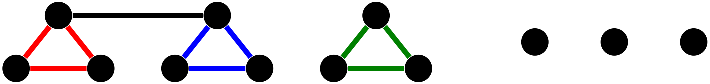

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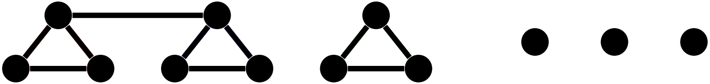

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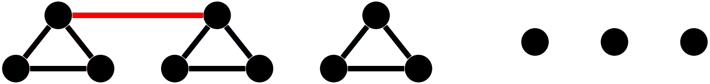

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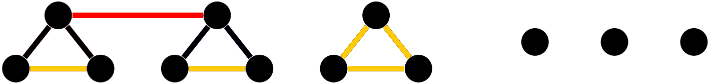

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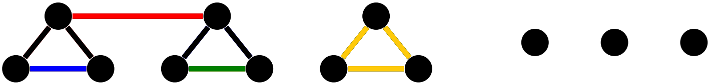

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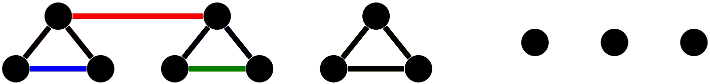

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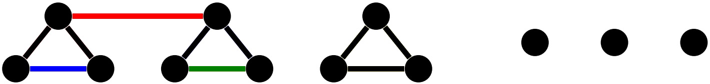

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G has n vertices, and $(m_1 K_2, \cdots, m_k K_2)$-saturated. Then it contains at least $3(m_1 + m_2 + \cdots + m_k - k)$ edges.

Suppose we have a counterexample G. G has less than $3(m_1 + m_2 + \cdots + m_k - k)$ edges and $(m_1 K_2, \cdots, m_k K_2)$-saturated. It has a coloring avoiding monochomatic matchings.

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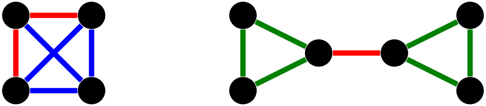
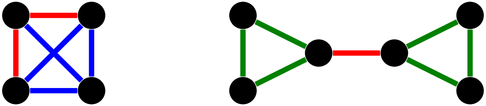


Lower bound

First, we change color of an edge into red as long as it doesn't create any $m_r K_2$.



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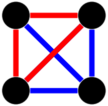

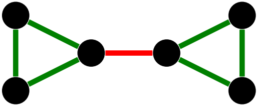

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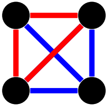

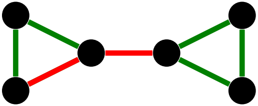

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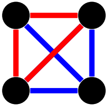

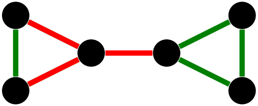

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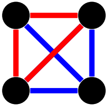

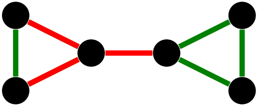

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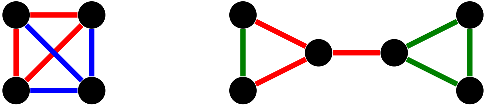

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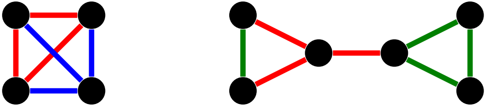

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We have this colored subgraph, then we start to change color of every edge into blue if possible to get a blue-heavy subgraph.

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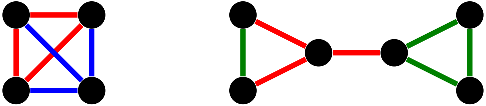

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Color heavy graphs

Theorem (Mader, 1973)

If G is mK_2-saturated, and $n \geq 2m - 1$, then one of the following holds.

1. Every component of G is an odd clique.

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Dominating vertices give too many edges, so G doesnot have any dominating vertex. Thus each color-heavy subgraph is disjoint union of odd cliques.
However, G may be not disjoint union of odd cliques, since some edge might change colors.

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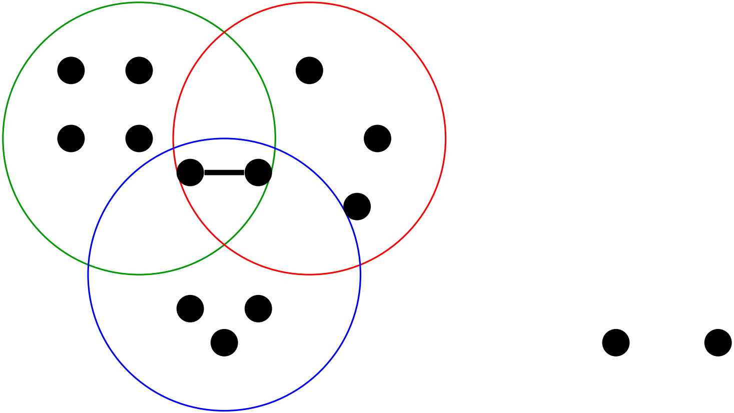
Lemma

Any edge e in G belongs to at most two color-heavy subgraph.



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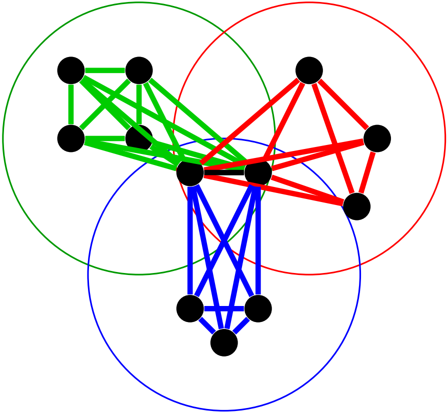

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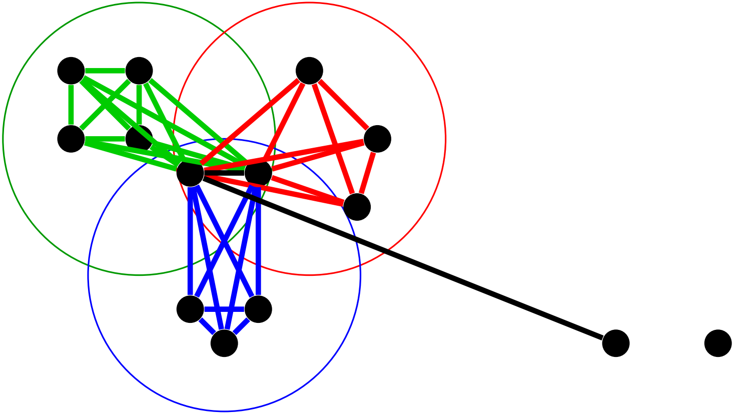
We add an edge, and recolor it.



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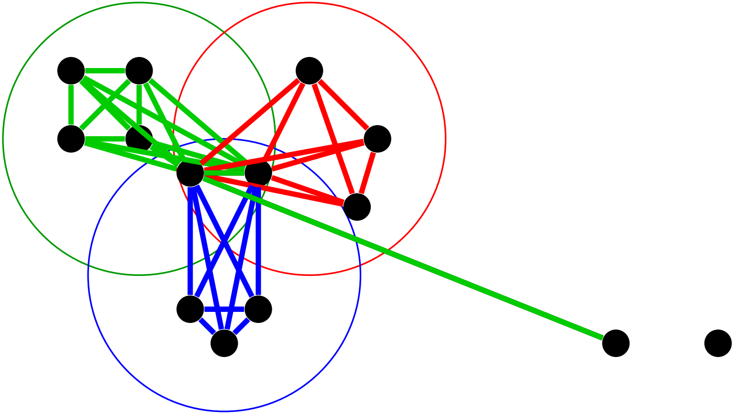

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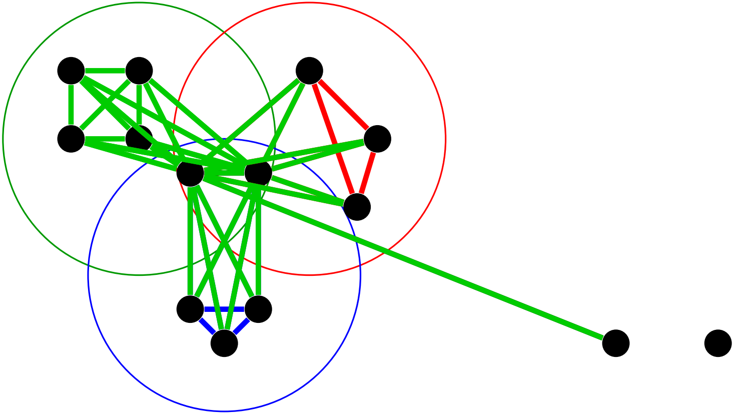

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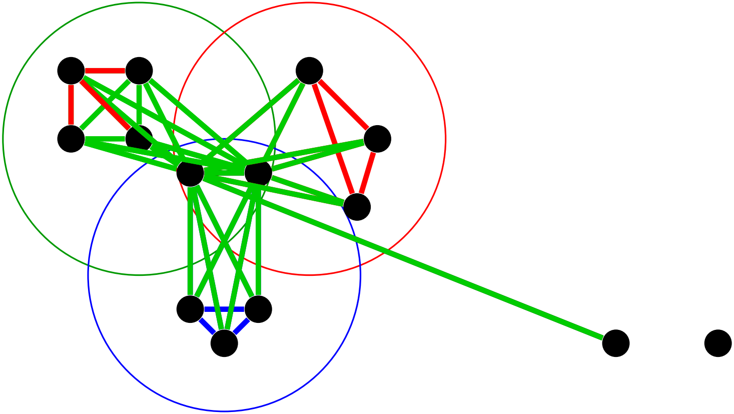

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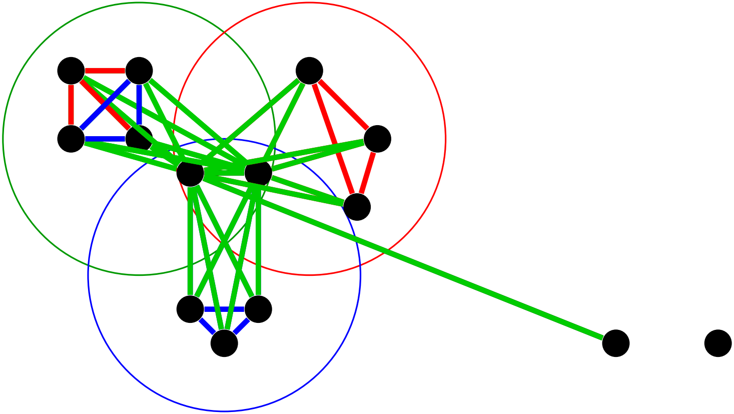

Lemma

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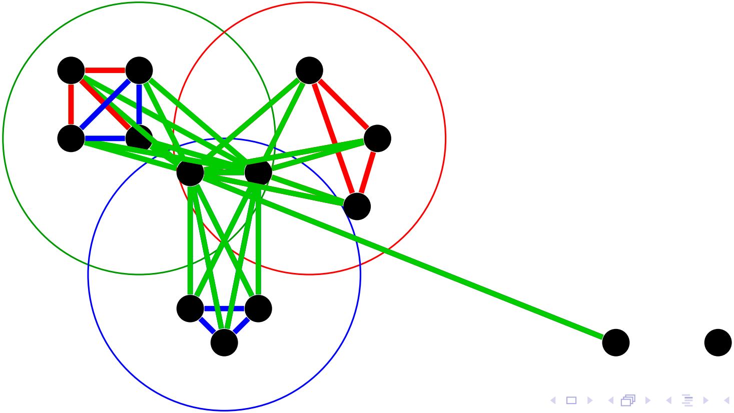
It does not contain a bigger matching, so G is not saturated, a contradiction.



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Each color-heavy subgraph is $K_{2t_{i,1}+1} \cup K_{2t_{i,2}+1} \cup \cdots \cup K_{2t_{i,s(i)}+1}$ with $t_{i,1} + t_{i,2} + \cdots + t_{i,s(i)} = m_i - 1$.

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For each color-heavy subgraph, we count 1 for each edge in K_3 or K_5 and we count $\frac{1}{2}$ for each edge in other components, then we count each edge in G exactly once.

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In addition, having equality means we have bunch of K_3 only.

If $m_i \geq 3$ for at least one i, they are vertex disjoint. If $m_i \leq 2$ for all i, they are just edge-disjoint.