Closure Operations and Forbidden Induced Subgraphs

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Closures and Forbidden Subgraphs

Basic Definitions and Notations

- graph $G = (V(G), E(G))$ ... simple, undirected
- $d_G(x)$ ... degree of a vertex x in a graph G
- circumference ... $c(G)$ ... the length of a longest cycle in G
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Closures - Motivation

Theorem: O.Ore (1960)

Let G be a connected graph of order $n \geq 3$ such that, for every pair of nonadjacent vertices x, y, $d_G(x) + d_G(y) \geq n$. Then G is hamiltonian.

Observation

If G is hamiltonian, then for every pair of nonadjacent vertices x, y, $G + \{xy\}$ is also hamiltonian.

- Question: Does the opposite implication hold???
- Answer: Not in general, but after adding some further conditions yes.

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- Closure operation - adding edges into G as long as possible respecting specified conditions.

Bondy-Chvátal's closure (1974) - based on the following lemma.

Lemma: O. Ore (1960)

Let G be a connected graph of order n, let x, y be a pair of nonadjacent vertices of G with $d_G(x) + d_G(y) \geq n$. Then G is hamiltonian **iff** $G' = G + \{xy\}$ is hamiltonian.

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Bondy-Chvátal's Closure

Definition of the Bondy-Chvátal's closure

Let G be a graph of order n. The last element of the sequence $G_1, \dots, G_t$ such that

1. $G_1 = G$,
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3. G_t contains no pair of nonadjacent vertices with $d_G(x) + d_G(y) \geq n$.

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Definition of a closure operator

Definition of eligibility

Let G be a graph and let H be a subgraph of G. We say that H is *P-eligible* in G, if H is not complete and satisfies some given condition or property P.

Notation - local completion

Let G be a graph and H a P-eligible subgraph of G. Then

$$B_H = \{xy;\, x, y \in V(H) \text{ and } xy \notin E(G)\}.$$

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is called *the P-closure of G* and is denote by $\mathrm{cl}_P(G)$.

In other words, the P-closure of a graph G is obtained from G by repeating a "local completion" at some P-eligible subgraph as long as possible.

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Closures of Graphs

Two basic questions:

1. Which conditions on eligibility of subgraphs allow us to add edges (complete eligible subgraph) into a graph in such a way that some specified graph property is preserved?
2. Which graph properties are preserved during the local completion of a graph at some eligible subgraph with specified condition (property) on eligibility?

First question

R. Kužel in his Ph.D. thesis characterized eligibility conditions of subgraphs, for which the local completion at eligible subgraph does not affect the circumference of graphs.

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Second question

Particular closure operators - preserves specified properties, lay stress on uniqueness.

Some closures:

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- H. Broersma & Z. Ryjáček (2001), preserves circumference,
- J. Brousek & P. Holub (2008), preserves circumference,
- Z. Ryjáček, L. Xiong, K. Yoshimoto, preserves existence of a 2-factor,
- Bollobás et. al. (1999), Z. Ryjáček & P. Vrána (2011), preserves hamiltonian connectedness,
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ANOTHER STORY

Forbidden Subgraphs - Definition

Definition

Let G be graph and $\mathcal{A}$ be a class of connected graphs. We say that G is $\mathcal{A}$-free if G contains none of the graphs from $\mathcal{A}$ as an induced subgraph.

Classes of graphs defined by forbidden subgraphs

- triangle-free graphs
- graphs with given girth
- line graphs
- claw-free graphs

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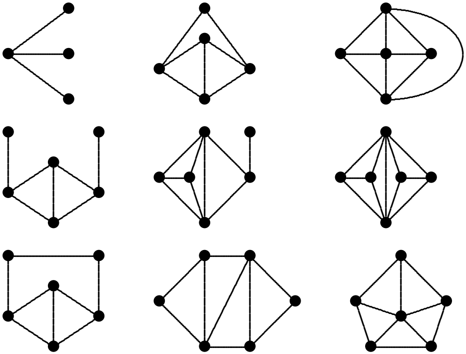
Line graphs and claw-free graphs

Theorem: Beineke (1969)

Let G be a graph. Then G is the line graph of a graph H ($L(H) = G$) **iff** G contains none of the following nine graphs as an induced subgraph.



Forbidden subgraphs for hamiltonicity

Observation

Let G be a 2-connected graph of order $n \geq 3$ and X a connected graph. Then G being X-free implies G is hamiltonian **iff** $X = P_3$.

Forbidden pairs with 2-connectivity

C, Z_1-free	Goodman, Hedetniemi	1974
C, N-free	Duffus, Gould, Jacobson	1981
C, Z_2-free	Gould, Jacobson	1982
C, P_6-free	Broersma, Veldman	1990

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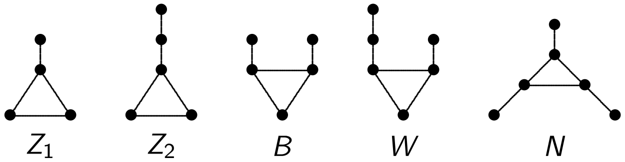
Forbidden subgraphs for hamiltonicity

Theorem: Bedrossian (1991)

Let X, Y be connected graphs such that $X, Y \neq P_3$, and let G be a 2-connected graph that is not a cycle. Then, G being X, Y-free implies G is hamiltonian **iff** (up to symmetry) $X = C$ and Y is an induced subgraph of one of P_6, Z_2, W, N.



Forbidden subgraphs for hamiltonicity

Forbidden pairs with 3-connectivity

C, Z_4-free, C, P_7-free, $C, N_{1,1,2}$-free	Brousek, Favaron, Ryjáček	1999
C, P_{11}-free	Luczak, Pfender	2004
C, Z_8-free	Lai, Xiong, Yan, Yan	2010

Forbidden triples for hamiltonicity - 2-connected

$C, N_{1,2,2}, N_{1,1,3}$-free, C, H, P_8-free, C, H, Z_5-free, $C, H, N_{1,1,4}$-free	Brousek, Ryjáček, Schiermeyer	1999
characterization C, X, Y-free	Brousek	2002
characterization without C	Faudree et al.	2002

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BOTH STORIES TOGETHER

Closures in claw-free graphs

Definition - Local connectivity

A vertex x is locally connected in a graph G if the set of all neighbours of x induces a connected graph in G. A graph G is locally connected if every vertex of G is locally connected.

Theorem: D. J. Oberly, D.P. Sumner (1979)

Every connected locally connected claw-free graph G is hamiltonian.

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Ryjáček's closure

Definition - eligible vertex

A vertex x is eligible in a graph G, if it is locally connected but its neighbourhood induces an incomplete graph.

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$B_x = \{uv;\ u, v \in N_G(x), uv \notin E(G)\}$.

Definition of the claw-free closure

Let G be a claw-free graph. The last element of a sequence $G_1, \ldots, G_t$ of graphs such that

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Ryjáček's closure - properties

Theorem: Ryjáček (1997)

Let G be a claw-free graph. Then

1. $\text{cl}(G)$ is uniquely determined,
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3. $c(G) = c(\text{cl}(G))$,
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Using this closure one can prove several sufficient condition involving forbidden subgraphs for hamiltonian properties!

Note that every connected locally connected claw-free graph has complete closure, hence the Oberly-Sumner's theorem can be obtained as a corollary of this statement.

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Ryjáček's closure - forbidden subgraphs

Definition - stability

A class of graphs $\mathcal{C}$ is stable under a closure clos, if $\forall G \in \mathcal{C}$ also $\text{clos}(G) \in \mathcal{C}$.

Note that the class of (non)hamiltonian graphs is stable under the claw-free closure.

Theorem: J. Brousek, Z. Ryjáček, I. Schiermeyer (1999)

A class of C, X-free graphs is stable under the claw-free closure **iff** $X \in \{P_i, Z_i, N_{i,j,k}, H, T\}$.

Theorem: Z. Ryjáček (2001)

Let G be a 2-connected graph of order $n \geq 11$. If G is C, X-free for $X \in \{P_6, Z_3, W, N\}$, then either $\text{cl}(G)$ is C, N-free or $\text{cl}(\text{G}) \in \mathcal{A}$.

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Edge closure of claw-free graphs

Based on local completion at eligible edges (locally connected with incomplete neighbourhood).

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Let G be a claw-free graph. Then

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There are more classes of graphs (defined by forbidden pairs) which are stable under the edge closure - trouble with their characterization.

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STORY NUMBER THREE

(last one, I promise)

Rainbow connection

Definition

A connected edge coloured graph G is rainbow-connected if any two distinct vertices are connected by a rainbow path, i.e., a path whose edges have pairwise distinct colours. The rainbow connection number $\mathrm{rc}(G)$ of G is the smallest number of colours such that G is rainbow connected.

Rainbow connection

Basic observation

- $rc(G) = 1$ iff G is complete
- $rc(G) = |V(G)| - 1$ iff G is a tree
- from previous statement $rc(G) \leq |V(G)| - 1$
- $rc(G) \geq \text{diam}(G)$
- $rc(C_k) = \lceil \frac{k}{2} \rceil$
- $rc(W_n) = \begin{cases} 1 \ldots n = 3 \\ 2 \ldots 4 \leq n \leq 6 \\ 3 \ldots n \geq 7 \end{cases}$
- $rc(K_{s,t}) = \min\{4, \lfloor \sqrt[s]{t} \rfloor\}$, for $2 \leq s \leq t$.

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- $rc(W_n) = \begin{cases} 1 \ldots n = 3 \\ 2 \ldots 4 \leq n \leq 6 \\ 3 \ldots n \geq 7 \end{cases}$
- $rc(K_{s,t}) = \min\{4, \lfloor \sqrt[s]{t} \rfloor\}$, for $2 \leq s \leq t$.

Rainbow connection

Basic observation

- $rc(G) = 1$ iff G is complete
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Rainbow connection and forbidden subgraphs

Question

For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph being $\mathcal{F}$-free implies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$?

Theorem: P. Holub, Z. Ryjáček, I. Schiermeyer (2013)

Let X be a connected graph. Then there is a constant k_X such than every connected X-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$ **iff** $X = P_3$.

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Final slide, but with a proof

Theorem: P. Holub, Z. Ryjáček, I. Schiermeyer (2013)

Let X, Y be connected graphs. Then there is a constant $k_{X,Y}$ such than every connected X, Y-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{X,Y}$ **iff** (up to symmetry) $X = C$ and $Y = N$, or $X = K_{1,r}$ $(r > 3)$ and $Y = P_4$.

Thank you for your attention!

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