University of Illinois, Urbana 2013

Hamiltonian Cycles in the Square of a Graph

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G^n .... n-**th power of** G: $V(G^n) = V(G)$

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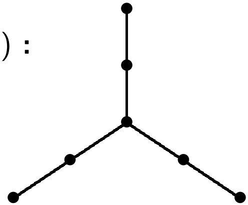
$S(K_{1,3})$ **:**



Square of the graph

Theorem

[Fleischner] *Let G be a 2-connected graph. Then G^2 is hamiltonian.*

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[Thomassen] *If the block graph of G is a path, then G^2 is hamiltonian.*

Connection of the graphs

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[Fleischner] *Let y and z be arbitrarily chosen vertices of a 2-connected graph G. Then G^2 contains a hamiltonian cycle C such that the edges of C incident with y are in G and at least one of edges of C incident with z is in G. If y and z are adjacent in G, then these are three different edges.*

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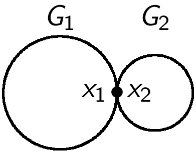

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c) Vertex **x** is **of type 3**: ... x is not of type 1 or 2 and $\exists$ C in G^2; there exist $u, v \in N_G(x)$ such that $uv \in E(C)$.

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$$V_{[i]}(G) = \{x \in V(G) \mid x \text{ is of type } i\}, \ \ i = 1, 2, 3, 4.$$

Theorem

Let G_1, G_2 *be connected graphs such that* $(G_1)^2$, $(G_2)^2$ *are hamiltonian, let* $x_i \in V(G_i)$, $i = 1, 2$. *If*

I) $G = G_1[x_1 = x_2]G_2$ *and* $x_i \in V_{[1]}(G_i) \bigcup V_{[2]}(G_i)$, $i = 1, 2$, *or*

II) $G = G_1[x_1 = x_2]K_2$, $x_1 \in V_{[1]}(G_1) \bigcup V_{[2]}(G_1) \bigcup V_{[3]}(G_1)$ *and* $V(K_2) = \{x_2, u\}$ *or*

III) $G = G_1[x_1 = x_2]G_2$, $x_1 \in V_{[3]}(G_1)$ *and* $x_2 \in V_{[1]}(G_2)$,

then G^2 *is hamiltonian.*

Moreover under the assumptions of I),

a) if $x_i \in V_{[1]}(G_i)$, $i = 1, 2$, then $x = x_1 = x_2 \in V_{[1]}(G)$;
b) if $x_1 \in V_{[1]}(G_1)$ and $x_2 \in V_{[2]}(G_2)$, then $x = x_1 = x_2 \in V_{[2]}(G)$;
c) if G_2 is 2-connected and $x_1 \in V_{[1]}(G_1) \bigcup V_{[2]}(G_1)$, then $v \in V_{[1]}(G)$ for any $v \in V(G_2)$, $v \neq x_2$;
d) if $x_i \in V_{[2]}(G_i)$, $i = 1, 2$, then $x = x_1 = x_2 \notin V_{[1]}(G) \bigcup V_{[2]}(G)$.

Moreover under the assumptions of II),

a) if $x_1 \in V_{[1]}(G_1)$, then $x = x_1 = x_2 \in V_{[1]}(G)$;
b) if $x_1 \in V_{[2]}(G_1)$, then $x = x_1 = x_2 \in V_{[2]}(G)$;
c) if $x_1 \in V_{[1]}(G_1) \bigcup V_{[2]}(G_1)$, then $u \in V_{[2]}(G)$.

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. $\rightarrow$ edge between cut vertex a and block B of G: $a \in V(B)$

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. $\rightarrow$ edge between cut vertex a and block B of G: $a \in V(B)$

Theorem

Let G be a graph such that its block graph is a path and let u be an arbitrary vertex which is not a cut vertex and is contained in an endblock of G. Then G^2 is hamiltonian and moreover

a) if u is contained in a cyclic block, then $u \in V_{[1]}(G)$,

b) if u is contained in an acyclic block, then $u \in V_{[2]}(G)$.

Main results

$$V_{\geq 3}(G) = \{x \in V(G) | d_G(x) \geq 3\}$$

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- for $x \in V(G)$, $t_G(x)$ denotes the number of acyclic non-end blocks of G containing x

Lemma

Let G be a connected graph with exactly one vertex in $V_{\geq 3}(Bl(G))$ corresponding to a cut vertex a of G. If a is contained in at most two acyclic non-end blocks of G, then G^2 contains a hamiltonian cycle C such that
if $t_G(a) = 0$, then both edges of C incident with a are in G,
if $t_G(a) = 1$, then exactly one edge of C incident with a is in G,
if $t_G(a) = 2$, then no edge of C incident with a is in G.

Theorem

Let G be a connected graph with at least three vertices such that

i) *every vertex $x \in V_{\geq 3}(Bl(G))$ corresponds to a cut vertex of G, and*
ii) *for any two vertices $x, y \in V_{\geq 3}(Bl(G))$, $dist_{Bl(G)}(x, y) \geq 4$.*

Then G^2 is hamiltonian if and only if every cut vertex of G is contained in at most two acyclic non-end blocks of G.

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The proof of Theorem is made by induction and Lemma is necessary in the induction hypothesis. The conditions in the theorem can be verified in polynomial time.

Corollary

Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center corresponds to a cut vertex a of G. Then the graph G^2 is hamiltonian if and only if the vertex a is contained in at most two acyclic non-end blocks of G.

Corollary

Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center corresponds to a cut vertex a of G. Then the graph G^2 is hamiltonian if and only if the vertex a is contained in at most two acyclic non-end blocks of G.

Corollary

If the block graph of G with at least three vertices is a star, then G^2 is hamiltonian.

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Denote B_c a block corresponding to a center of a star

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Theorem

Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center corresponds to a block B_c of G. If $(B_c)^2$ contains an acceptable cycle in G, then G^2 is hamiltonian.

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Theorem

[Schaar] *For every block G with $|V(G)| \geq 4$ it holds that there exists a hamiltonian cycle in G^2 such that it contains at least four edges of G.*

Conjecture: *Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center c corresponds to a block B_c of G. If $d_{Bl(G)}c \leq k$, $k < 5$, then G^2 is hamiltonian.*

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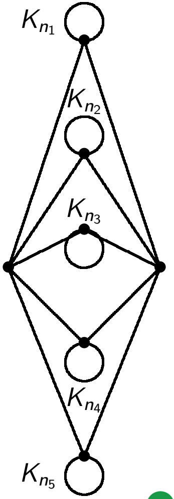
Conjecture is true for $k \leq 2$ but for $k = 3, 4$ is an open problem.


K_{n_1}
K_{n_2}
K_{n_3}
K_{n_4}
K_{n_5}

Conclusion

Thank you for your attention.

This contribution was supported by the European Regional Development Fund (ERDF), project NTIS - New Technologies for the Information Society, European Centre of Excellence, CZ.1.05/1.1.00/02.0090.

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