Coloring a claw-free graph with Δ-1 colors

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Joint with Landon Rabern

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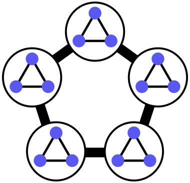

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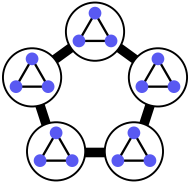

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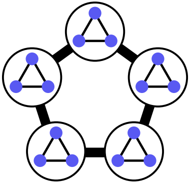

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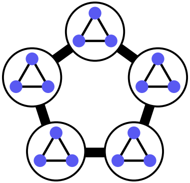

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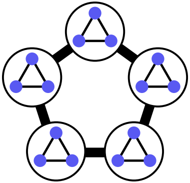

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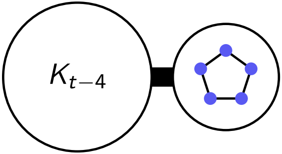

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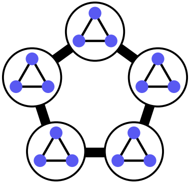

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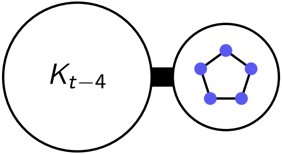

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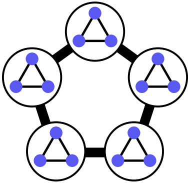

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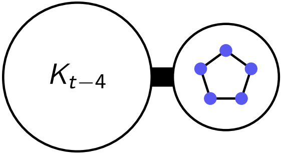

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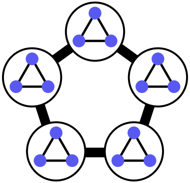

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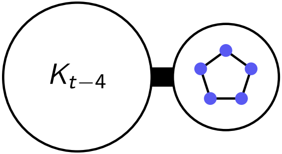

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- Forbidden subgraphs via list-coloring

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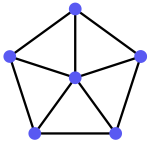

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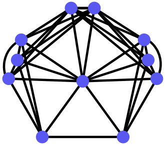

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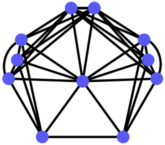

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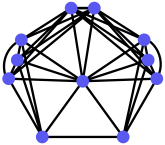

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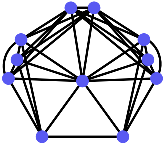

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For any graph G and function f (such that $f(v) < |G|$ for all $v \in G$), to verify that G is f-choosable, it suffices to consider all list assignments L such that $|L(v)| = f(v)$ and $|\cup_{v \in G} L(v)| < |G|$.

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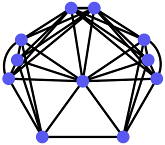

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Key Idea: No d_1-choosable graph can appear as an induced subgraph in a minimal counterexample to B-K Conj.

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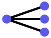
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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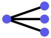
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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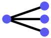
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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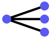
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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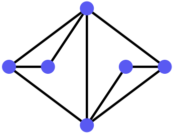

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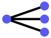
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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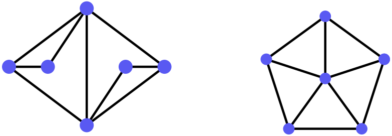

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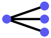
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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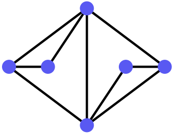
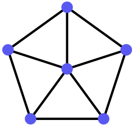
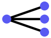




First Step: B-K Conj. is true for quasi-line graphs.

Main Result

Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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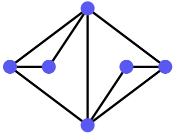

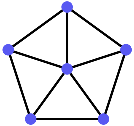

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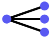
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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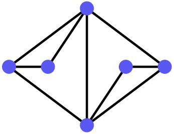

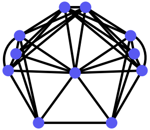

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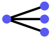
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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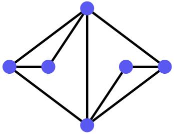

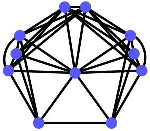

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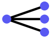
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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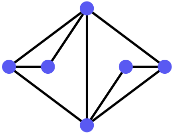

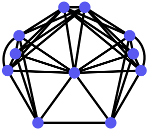

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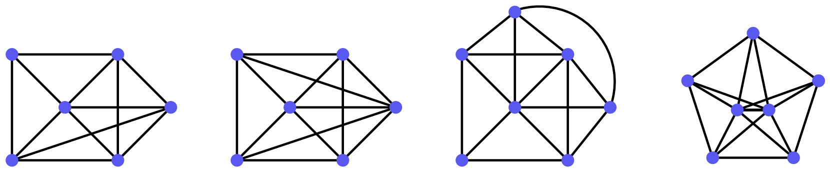
Gallery of d_1-choosble graphs



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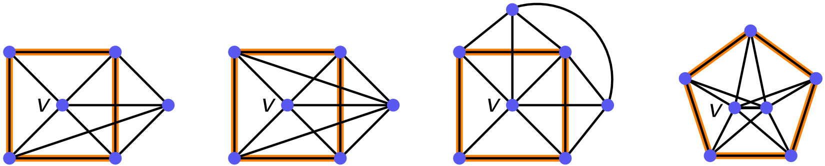

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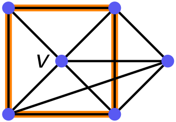
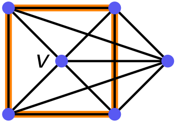
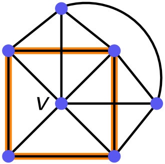


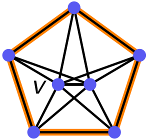
D_6



$C_4 \vee K_2$



D_7



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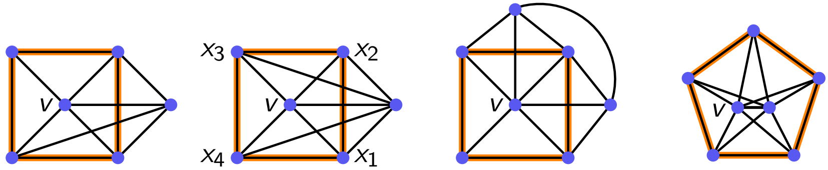

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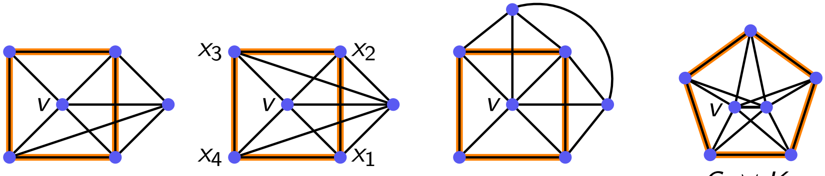

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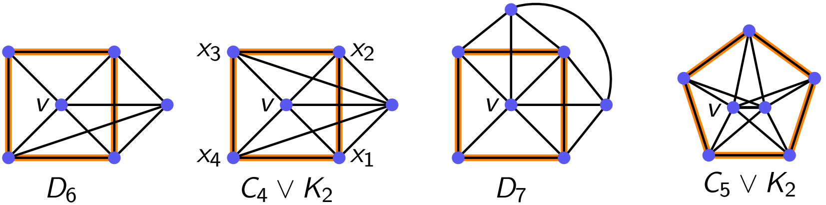

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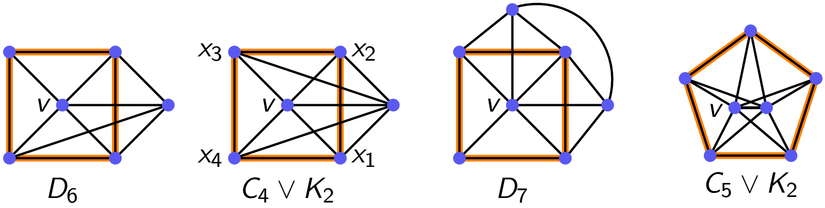

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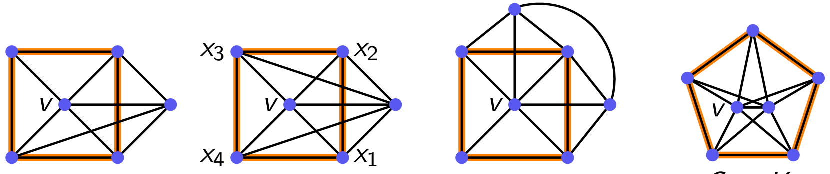

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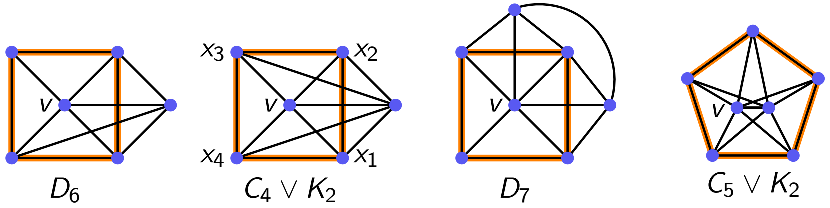

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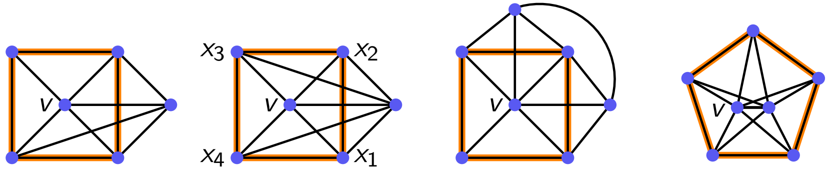

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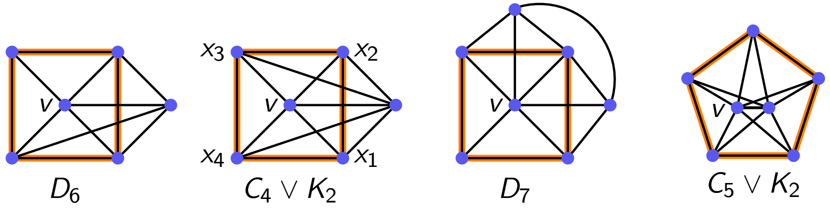

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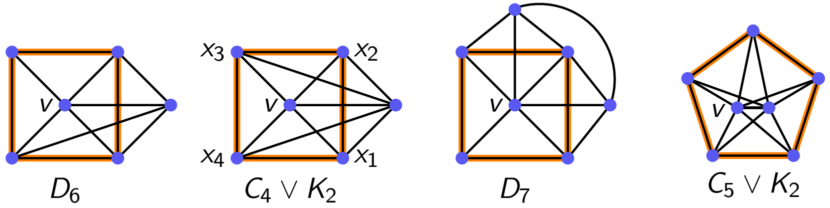

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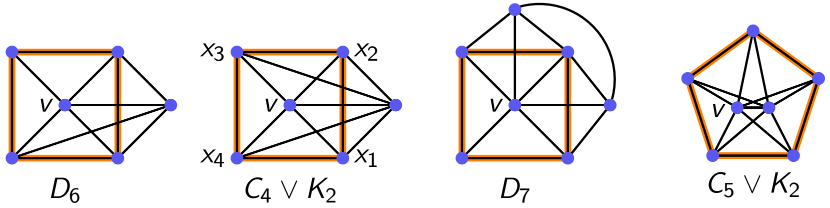

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- Else color 4-cycle with c and $d \in (L'(x_1) \cup L'(x_3)) \setminus L'(v)$
  (and some other color), then finish.

Key Lemma (Outline)

Key Lemma: If G is claw-free, but not quasi-line, and G is a minimal counterexample to the B-K Conjecture, then G contains a vertex v such that $N(v)$ is a thickening of C_5.

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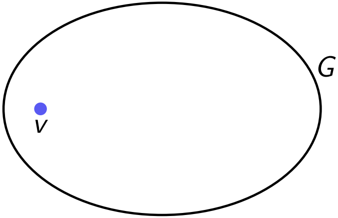

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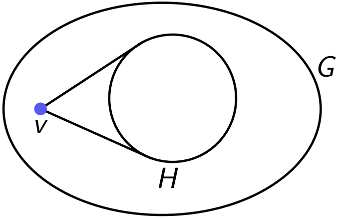

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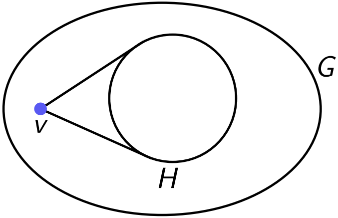

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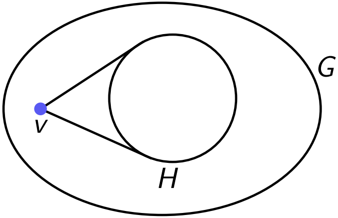

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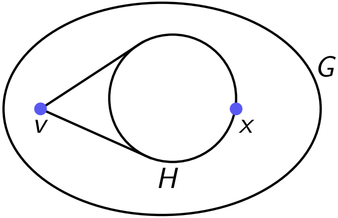

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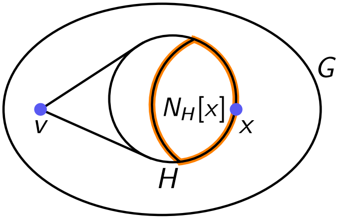

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Say instead H has an induced C_4, $x_1x_2x_3x_4$.

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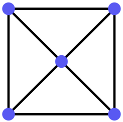

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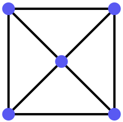

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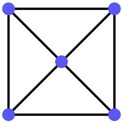

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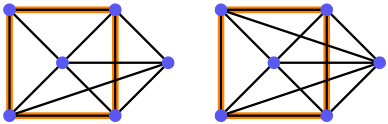

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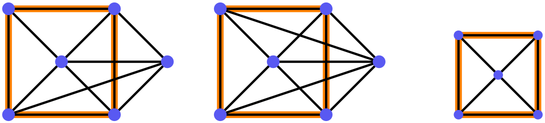

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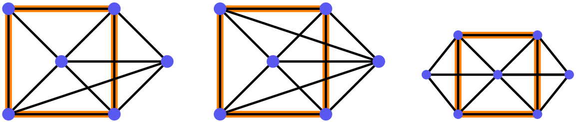

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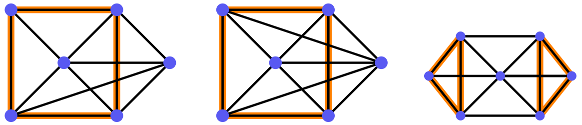

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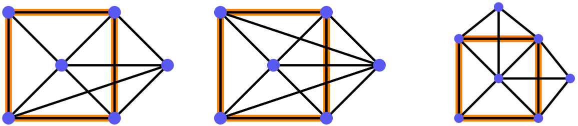

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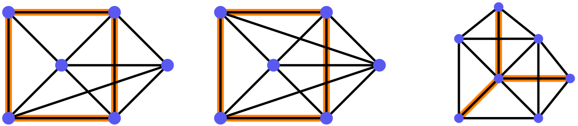

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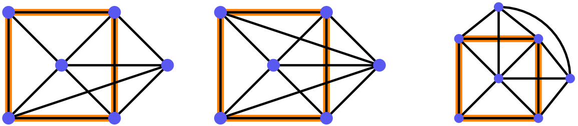

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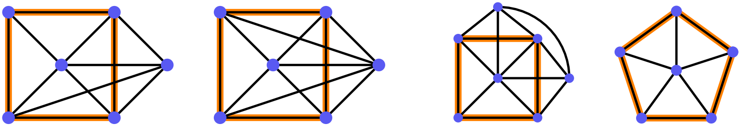

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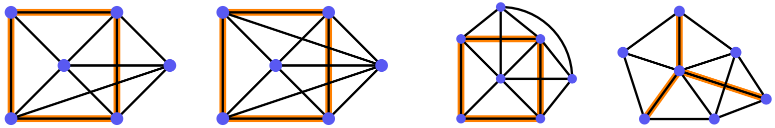

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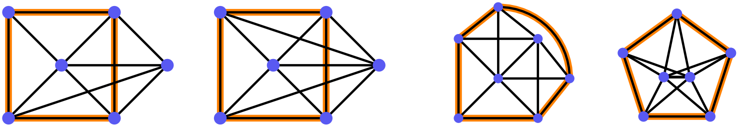

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So H contains a C_5. Each other neighbor y of v must be adj. to at least 3 succesive verts on the C_5 or we get a claw. If y is adj. to 4 or 5 cycle verts, we get a d_1-choosable subgraph.

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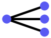
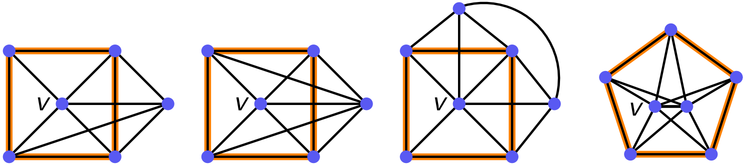
Summary

Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

- **First Step:** B-K Conj. is true for quasi-line graphs.
- **Key Lemma:** If G is claw-free, but not quasi-line, and G is a minimal counterexample to the B-K Conjecture, then G contains a vertex v such that $N(v)$ is a thickening of C_5.
- **Key Idea:** A minimal counterexample to B-K Conjecture cannot contain a d_1-choosable graph as an induced subgraph.



- **Final Step:** Since G is claw-free, nbrs of verts in thickening attach in a structured way, so we get a d_1-choosable subgraph.