The Manickam-Miklós-Singhi Conjecture for Vector Spaces

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Toy Problem

Question

Over all choices of n real numbers with nonnegative sum, what is the minimum number of subsets of these n real numbers that must also have nonnegative sum?

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Answer

$$2^{n-1}$$

Extremal Construction

$$x_1 = n-1, x_2 = -1, \ldots, x_n = -1$$

Restricting the Size of the Subsets

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Over all choices of n real numbers with nonnegative sum, what is the minimum number of k-element subsets of these n real numbers that must also have nonnegative sum?

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Same Example

$$x_1 = n-1, x_2 = -1, \ldots, x_n = -1$$

shows that the answer is at most $\binom{n-1}{k-1}$.

Restrictions on n

If n is large enough, is $\binom{n-1}{k-1}$ the answer?

The Manickam-Miklós-Singhi Conjecture

Conjecture [Manickam-Miklós-Singhi, 1988]

If $n \geq 4k$, then every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets with nonnegative sum.

Much Interest

N. Alon, H. Aydinian, A. Bhattacharya, T. Bier, V.M. Blinovsky, G. Chiaselotti, A. Chowdhury, P. Frankl, S. Hartke, H. Huang, G. Infante, N. Manickam, G. Marino, D. Miklós, A. Pokrovskiy, G. Sarkis, S. Shahriari, N.M. Singhi, D. Stolee, B. Sudakov, M. Tyomkyn.

What's Known?

Theorem [Bier-Manickam, 1987]

MMS Conjecture true if $k|n$.

Theorem [Alon-Huang-Sudakov, 2012]

MMS Conjecture true if $n \geq \min\{33k^2, 2k^3\}$.

Recent Arxiv Annoucements

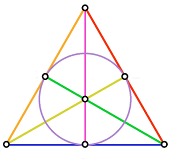
- MMS Conjecture true if $n \geq 10^{46}k$ [Pokrovskiy, Aug 9 2013].
- MMS Conjecture is true [Blinovsky, Oct 4 2013].

Theorem [Manickam, 1986; Marino-Chiaselotti, 2002; C, 2014]

MMS Conjecture true if $k = 3$.

Vector Space Analogues

Fano Plane

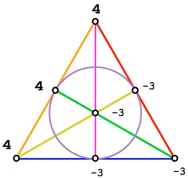


- Fano Plane represents a 3-D vector space over $\mathbb{F}_2$.
- Seven 1-D subspaces represented by points.
- Seven 2-D subspaces represented by lines.

Weights

Weights of 1-D spaces

Assign real-valued weights to the 1-D subspaces of the Fano plane in such a way that the sum of the weights is nonnegative.



Weight of a general subspace

Define the weight of a subspace to be the sum of the weights of the 1-D subspaces it contains.

Same Question

Question for the Fano Plane

Over all choices of real-valued weights for the 1-D subspaces with nonnegative sum, what is the minimum number of 2-D subspaces with nonnegative weight?

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Question for the Fano Plane

Over all choices of real-valued weights for the 1-D subspaces with nonnegative sum, what is the minimum number of 2-D subspaces with nonnegative weight?

Answer for the Fano Plane

$$1$$

General Question

Question

Let V be an n-D vector space over a finite field. Over all choices of real-valued weights for the 1-D subspaces of V with nonnegative sum, what is the minimum number of k-D subspaces of V with nonnegative weight?

The Gaussian Binomial Coefficients

Definition

For a positive integers q and n, define

$$[n]_q := \frac{q^n-1}{q-1} = q^{n-1} + q^{n-2} + \cdots + 1.$$

Definition

Define $[n]_q!$ by $[0]_q! = 1$ and $[n]_q! = [n]_q[n-1]_q!$

Definition

For a positive integer k and $n \geq k$, define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{0 \leq i < k} \frac{q^{n-i}-1}{q^{k-i}-1}.$$

Enumeration

What does $\begin{bmatrix} n \\ k \end{bmatrix}$ count?

- $\begin{bmatrix} n \\ k \end{bmatrix}$ counts the number of k-D subspaces of V.
- $\binom{n}{k}$ counts the number of k-element subsets of $\{1, \ldots, n\}$.

What does $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ count?

- $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ counts the number of k-D subspaces of V containing a fixed 1-D subspace.
- $\binom{n-1}{k-1}$ counts the number of k-element subsets of $\{1, \ldots, n\}$ containing a fixed element.

The Manickam-Miklós-Singhi Conjecture for Vector Spaces

Setup

Let V be an n-D vector space over $\mathbb{F}_q$. Assign real-valued weights to the 1-D subspaces of V so that their sum is nonnegative. Define the weight of a subspace $S \subset V$ to be the sum of the weights of the 1-D subspaces it contains.

Conjecture [Manickam-Singhi, 1988]

If $n \geq 4k$, then there are at least $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ k-D subspaces of V with nonnegative weight.

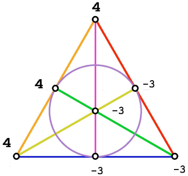
Theorem [Manickam-Singhi, 1988]

True when $k|n$.

Why $n \geq 4k$?

$n = 3,\ k = 2$

Notice $1 < \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3$.



Counterexamples

When $k + 1 < n < 2k - 1$, there are real-valued weights for the 1-D subspaces of V with nonnegative sum such that there are fewer than $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ k-dimensional subspaces with nonnegative weight.

MMS Conjecture for Vector Spaces is True

Setup

Let V be an n-D vector space over $\mathbb{F}_q$. Assign real-valued weights to the 1-D subspaces of V so that their sum is nonnegative. Define the weight of a subspace $S \subset V$ to be the sum of the weights of the 1-D subspaces it contains.

Theorem [C-Sarkis-Shahriari, 2013+; Huang-Sudakov, 2013+]

- If $n \geq 3k$, then there are at least $\left[{n-1 \atop k-1}\right]$ k-D subspaces with nonnegative weight.
- If equality holds, the family of all k-D subspaces with nonnegative weight is a star on a fixed 1-D subspace.

Back to Sets

Original Question

Over all choices of n real numbers with nonnegative sum, what is the minimum number of k-element subsets of these n real numbers that must also have nonnegative sum?

Inclusion Matrices

Definition

The inclusion matrix W_{jk} is an $\binom{n}{j} \times \binom{n}{k}$ matrix whose

- rows are indexed by the j-subsets of $\{1, \ldots, n\}$
- columns are indexed by the k-subsets of $\{1, \ldots, n\}$
- entry in row Y and column S is 1 if $Y \subset S$ and is 0 otherwise.

Example: $n = 3$, $j = 1$, $k = 2$.

$$\begin{array}{c} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \begin{array}{c} \begin{array}{ccc} \{1,2\} & \{1,3\} & \{2,3\} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

Weights

Vector of Weights

Given n real numbers $x_1, \ldots, x_n$, define the vector

$$\vec{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T \in \mathbb{R}^n.$$

Weights of k-element sets

The entries of $W_{1k}^T \vec{x}$ give the weights of the k-element sets.

$$\begin{array}{c} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \end{array} \begin{array}{c} \begin{array}{ccc} \{1\} & \{2\} & \{3\} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{c} \{1\} \\ \{2\} \\ \{3\} \end{array} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{array}{c} \{1,2\} \\ \{1,3\} \\ \{2,3\} \end{array} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Bose-Mesner Matrices

Assumption

Can assume that $\vec{x} \neq \vec{0}$ and that $\langle \vec{x}, \vec{1} \rangle = 0$.

Theorem [Wilson, 1984]

If $\vec{x} \neq \vec{0}$ and $\langle \vec{x}, \vec{1} \rangle = 0$, then the vector $W_{1k}^T \vec{x}$ is an eigenvector of each of the Bose-Mesner matrices $B_0, \ldots, B_k$.

Definition

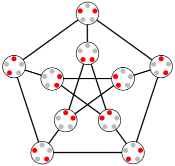
The Bose-Mesner matrix B_j is an $\binom{n}{k} \times \binom{n}{k}$ matrix whose

- rows and columns are indexed by the k-subsets of $\{1, \ldots, n\}$.
- $B_j(S, T) = \binom{k - |S \cap T|}{j}$.

B_k is adjacency matrix of Kneser Graph

Kneser Graph

- Vertices labeled with the k-subsets of $\{1, \ldots, n\}$
- $S \sim T$ iff $|S \cap T| = 0$.



B_k

$$B_k(S, T) = \binom{k - |S \cap T|}{k} = \begin{cases} 1 & \text{if } |S \cap T| = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Eigenvectors

Theorem [Wilson, 1984]

If $\vec{x} \neq \vec{0}$ and $\langle \vec{x}, \vec{1} \rangle = 0$, then the vector $W_{1k}^T \vec{x}$ is an eigenvector of the Bose-Mesner matrix B_j with eigenvalue

$$-\binom{k-1}{j-1}\binom{n-j-1}{k-1}$$

for $0 \leq j \leq k$.

Why are Eigenvectors Useful?

Because they allow us to crowd a lot of nonnegative k-sets on a single point.

An Illustration

Algebraic Techniques for Sets and Vector Spaces

- Recall MMS Conjecture is true when $n \geq \min\{33k^2, 2k^3\}$ [Alon-Huang-Sudakov, 2012].
- Using eigenvectors, we give a different proof that MMS Conjecture is true when $n \geq 2k^3$.
- Same techniques prove MMS Conjecture for Vector Spaces.

Part I: Eigenvectors

Decreasing Order

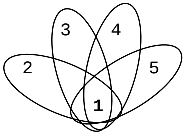
Can assume $x_1 \geq \cdots \geq x_n$.

Many nonnegative k-sets on x_1

Use eigenvectors to crowd nonnegative k-sets on x_1.

Done if ...

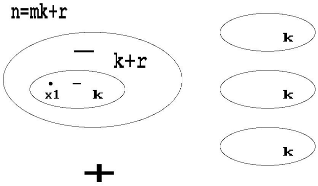
If all k-sets on x_1 are nonnegative, then done.



Part II: Perfect Matchings

A negative k-set on x_1

- May assume there is a negative weight k-set on x_1.
- Permuting perfect matchings yields many nonnegative k-sets off x_1.
- So many that there are greater than $\binom{n-1}{k-1}$ nonnegative k-sets when $n \geq 2k^3$.



Using the Kneser Graph

Highest Weight k-Set

$A = \{x_1, \ldots, x_k\}$ is the highest weight k-set.

Eigenvector

Inner product of the row of B_k corresponding to A and $\vec{b} := W_{1k}^T \vec{x}$:

$$\sum_{|S \cap A| = 0} b_S = A \overset{B_k}{\begin{pmatrix} * & * & * \end{pmatrix}} \overset{\vec{b}}{\begin{pmatrix} * \\ * \\ * \end{pmatrix}} = -\binom{n-k-1}{k-1} b_A.$$

Crowding Nonnegative k-sets on A

Negate

Since the sum of the weights of all k-sets is zero,

$$\sum_{|S \cap A| \neq 0} b_S = \binom{n-k-1}{k-1} b_A.$$

Highest Weight

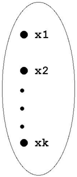
Since A is highest weight k-set, at least $\binom{n-k-1}{k-1}$ nonnegative k-sets intersecting A.

Crowding Nonnegative k-sets on x_1

Nonoptimal But Suffices for Cubic

- Since x_1 is largest real, if a nonnegative k-set contains x_i but not x_1, can swap x_1 for x_i to get a nonnegative k-set on x_1.
- Hence, number of nonnegative k-sets on x_1 is at least

$$\frac{1}{k}\binom{n-k-1}{k-1}.$$



A Negative $(k+r)$-set on x_1

Negative k-set on x_1

May assume there is a negative k-set T on x_1; else done.

Division

Write $n = mk + r$, where $0 \leq r \leq k-1$.

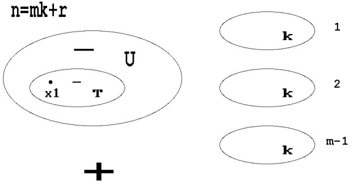
Induction

$$\sum_{T \subset S \in \binom{[n]}{k+1}} \left(\sum_{x_i \in S} x_i \right) = (n-k)b_T + \sum_{x_i \notin T} x_i = (n-k-1)b_T < 0.$$

Repeat until you get a negative $(k+r)$-set U containing T.

Permute and Double-Count

Counting Nonnegative k-sets off x_1



By permuting and double-counting, at least

$$\binom{n-(k+r)-1}{k-1} \geq \binom{n-2k}{k-1}$$

nonnegative k-sets disjoint from U and hence T and x_1.

MMS Conjecture true if $n \geq 2k^3$

Finishing Cubic

- If there is a negative k-set on x_1, then when $n \geq 2k^3$, number of nonnegative k-sets is at least

$$\underbrace{\frac{1}{k}\binom{n-k-1}{k-1}}_{\text{nonneg on } x_1} + \underbrace{\binom{n-2k}{k-1}}_{\text{nonneg off } x_1} > \binom{n-1}{k-1}.$$

- Hence, extremal configuration is a star.
- Same argument proves MMS Conjecture for Vector Spaces.

Additional Improvements with More Eigenvalues

Using the Bose-Mesner matrix B_k

$$\sum_{|S \cap A|=0} b_S = -\binom{n-k-1}{k-1} b_A.$$

Using the Bose-Mesner matrix B_{k-1}

$$\sum_{|S \cap A|=1} b_S = \left(\binom{n-k-1}{k-1} - (k-1)\binom{n-k-1}{k-2}\right) b_A$$

$$\geq \left(1 - \frac{1}{c}\right)^2 \binom{n-1}{k-1} b_A \text{ if } n \geq ck^2.$$

Another High Weight k-Set

Claim

$C = \{x_1, x_{k+1}, \dots, x_{2k-1}\}$ has high weight with respect to A if there are most $\binom{n-1}{k-1}$ nonnegative k-sets.

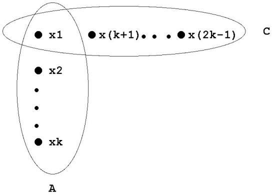
Reason

C is the highest weight k-set that intersects A in exactly 1 point:

$$b_C \geq \frac{\sum_{|S \cap A|=1} b_S}{\binom{n-1}{k-1}} \geq \frac{\left(1 - \frac{1}{c}\right)^2 \binom{n-1}{k-1} b_A}{\binom{n-1}{k-1}} = \left(1 - \frac{1}{c}\right)^2 b_A.$$

Nonnegative k-Sets Intersecting A and C

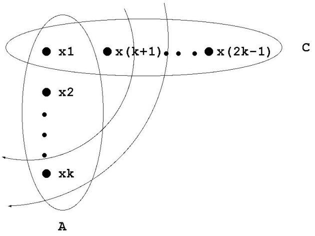
Crowding Nonnegative k-Sets onto A and C



- Many nonnegative k-sets intersecting A.
- Many nonnegative k-sets intersecting C.
- But if many nonnegative k-sets intersect C but not A, then greater than $\binom{n-1}{k-1}$ nonnegative k-sets.
- Hence, many nonnegative k-sets intersect A and C.

Crowding Nonnegative k-Sets on x_1

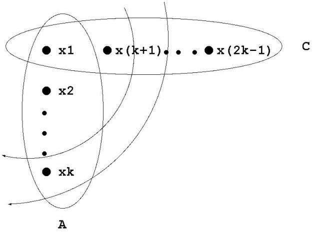
Subtract Nonnegative k-Sets off x_1



$$|\text{Nonnegative } k\text{-sets intersecting } A \text{ and } C \text{ but not on } x_1|$$
$$\leq |k\text{-sets intersecting } A \text{ and } C \text{ but not on } x_1|.$$

Obstacle to Getting Quadratic for Sets

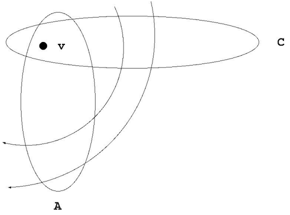
Subtract Nonnegative k-Sets off x_1



When $n \geq ck^2$, number of k-sets intersecting A and C but not on x_1 is too large with respect to $\binom{n-1}{k-1}$.

Proof for Vector Spaces when $n \geq 3k$

Subtract Nonnegative k-Sets off x_1



If A and C are two k-D subspaces in V that intersect in exactly a 1-D subspace v, then the number of k-D subspaces that intersect A and C but don't contain v is at most

$$\frac{1}{q^{n-3k}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Erdős-Ko-Rado Theorem

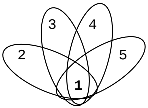
Definition

A family $\mathcal{F} \subset \binom{[n]}{k}$ is **intersecting** if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$.

Theorem [Erdős-Ko-Rado, 1961]

If $n \geq 2k$ an intersecting family $\mathcal{F} \subset \binom{n}{k}$ satisfies

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$



The Manickam-Miklós-Singhi Conjecture

Conjecture [Manickam-Miklós-Singhi, 1988]

If $n \geq 4k$, then every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets with nonnegative sum.

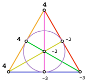
MMS for Vector Spaces when $2k < n < 3k$.

Theorem [C-Sarkis-Shahriari, 2013+; Huang-Sudakov, 2013+]

MMS Conjecture for Vector Spaces is true if $n \geq 3k$.

Counterexamples

When $k+1 < n < 2k-1$, there are real-valued weights for the 1-D subspaces of V with nonnegative sum such that there are fewer than $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ k-dimensional subspaces with nonnegative weight.



Open Problem

Is MMS Conjecture for Vector Spaces true when $2k < n < 3k$?