The List Version of the Borodin–Kostochka Conjecture for Graphs with Large Max Degree

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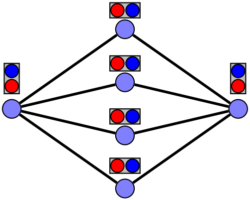

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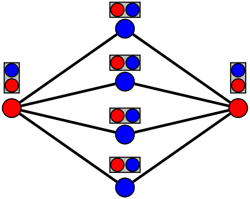

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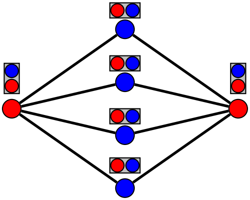

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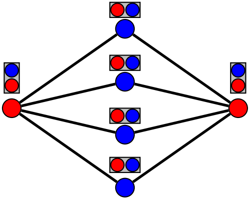

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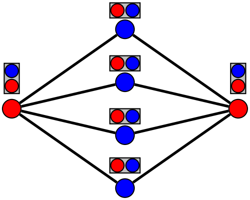

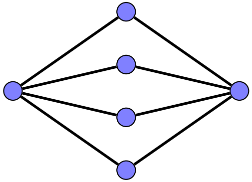

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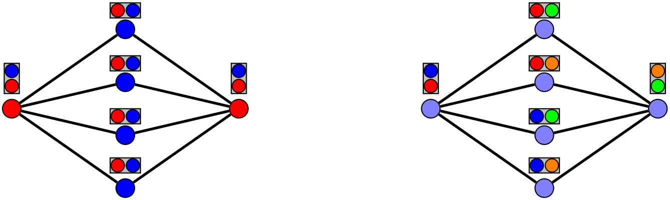

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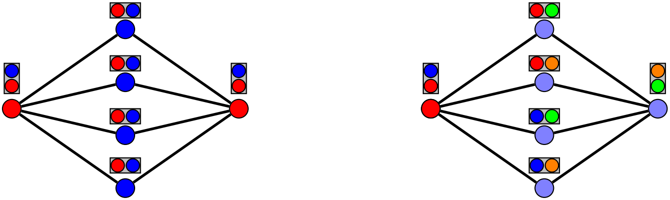

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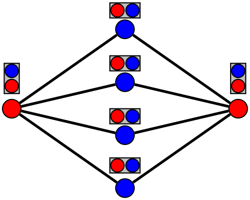

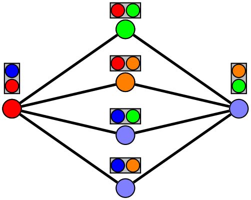

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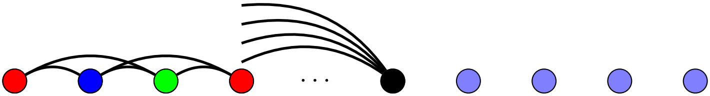

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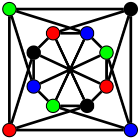
Conjecture (Reed 1998)

For any graph G,

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil$$



$$\frac{2+4+1}{2} < 4 \leq \left\lceil \frac{2+4+1}{2} \right\rceil$$

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Can this be extended?

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Conjecture (Borodin–Kostochka 1977)

Given a graph G with $\Delta(G) \geq 9$*,*

$$\text{if } \omega(G) \leq \Delta(G) - 1, \text{ then } \chi(G) \leq \Delta(G) - 1$$

If true, then sharp. Different phrasing:

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Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 9$ *contains a* $K_{\Delta(G)}$*.*

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Theorem (Reed 1999)

Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 10^{14}$ contains a $K_{\Delta(G)}$.

Reed claims the thm is still true for max degree at least 10^3, but not 10^2.

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Not true even for the ordinary chromatic number when $k = 2$.

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Given a graph G with $\Delta(G) \geq 10^{20}$,

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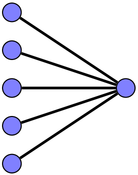

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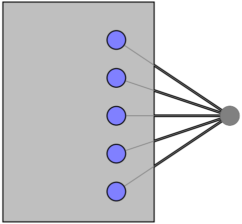

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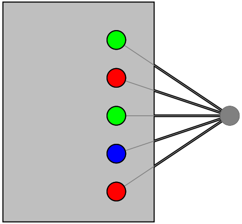

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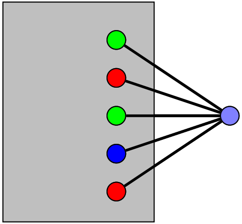

1. Decompose G "nicely" so we can analyze smaller pieces.

Theorem (Lovász Local Lemma)

Consider a set of (bad) events $\mathcal{E}$ where each E satisfies the following:

- $Pr(E) \leq p$
- *E is mutually independent to a set of all but at most d other events.*

If $ep(d+1) \leq 1$, then with positive probability, no event in $\mathcal{E}$ occurs.

2. Color randomly to obtain a good (enough) partial coloring.

Theorem (Azuma's Inequality)

Let X be a random variable determined by n trials $T_1, \ldots, T_n$. For each i and any two possible outcomes, if the following holds:
$|E(X|T_1 = t_1, \ldots, T_i = t_i) - E(X|T_1 = t_1, \ldots, T_i = t_i')| \leq c_i$,
then $Pr(|X - E(X)| > t) \leq 2e^{-t^2/(2\sum c_i^2)}$.

3. For the remaining graph, color greedily to show that G cannot exist.

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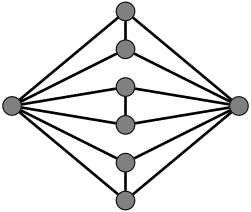
1. Randomly choose a color in $L(v)$ to use on v.
2. Remove any conflicts.



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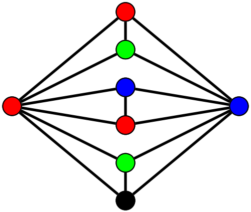

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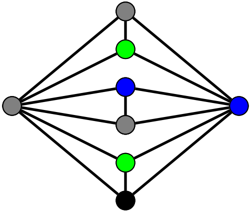

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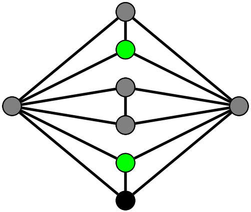

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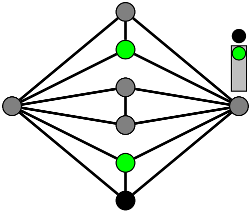
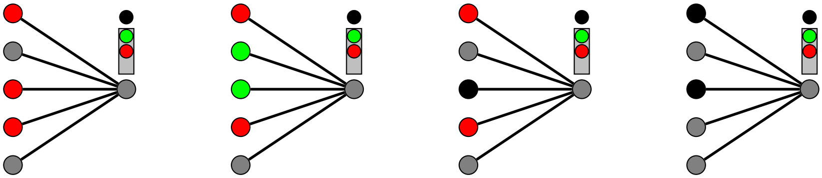


Definition

Given a partial coloring of G, an uncolored vertex v of degree Δ is *safe* if one of the following occurs:

- a color is repeated three times in $N(v)$;
- two colors are repeated twice in $N(v)$;
- a color is repeated twice in $N(v)$ and a color not in $L(v)$ is in $N(v)$;
- two colors not in $L(v)$ appear in $N(v)$.



Note that a vertex with two uncolored neighbors can always be colored.

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Lemma

If C is a clique of G, then there exists $C' \subset C$ such that $|C'| = |C| - 1$ and $|L(x) \cap L(y)| \geq |C| - 3$ for all $x, y \in C'$.

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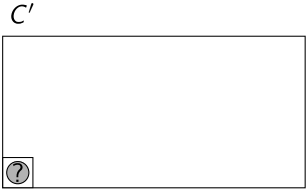

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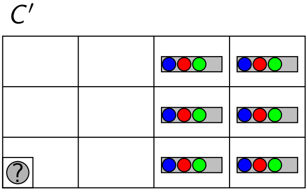

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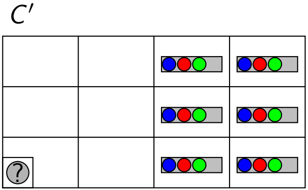

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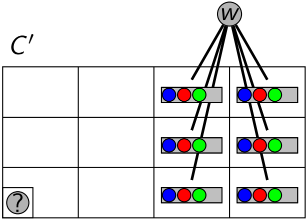

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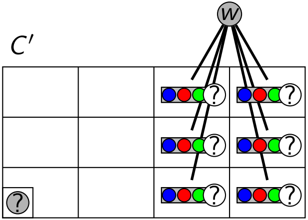

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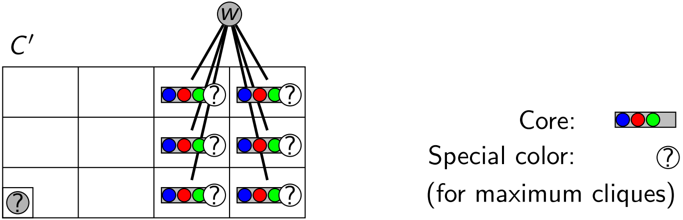

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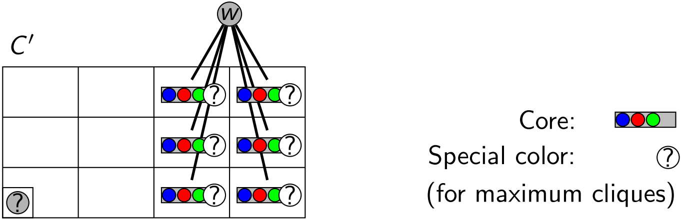


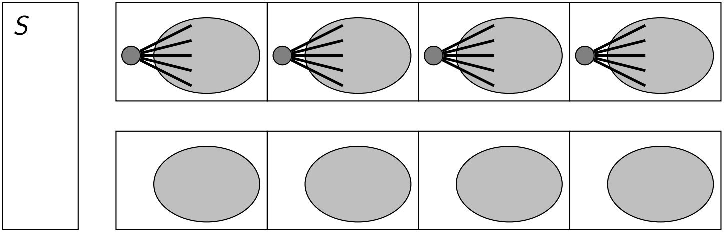
- Each vertex in a big clique C has at most one neighbor outside of C with more than 4 neighbors in C.
- There is at most one vertex outside of a $(\Delta - 1)$-clique with more than 4 neighbors in the clique.
- Each special color appears in at most 5 lists.

A vertex v is *sparse* if $||N(v)|| < \binom{\Delta}{2} - o(\Delta^2)$.

Lemma

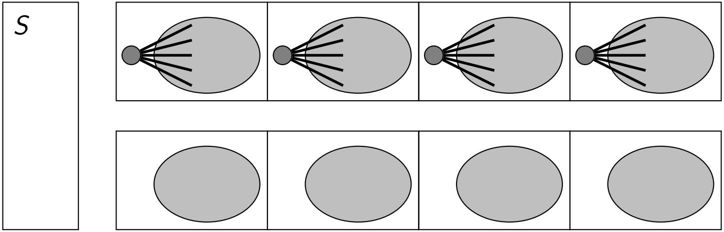
We can partition $V(G)$ into sets $S, D_1, \ldots, D_l$ such that

- *every vertex in S is sparse;*
- *$\exists w_i \in D_i$ such that $D_i - w_i$ is a clique of size $\Delta - o(\Delta)$;*
- *no vertex outside of D_i has more than $\frac{3\Delta}{4}$ neighbors in D_i and w_i has at least $\frac{3\Delta}{4}$ neighbors in D_i*



Partition the cliques further.

1. $\mathcal{P}_3$: set of maximum cliques C where some vertex not in C has "many" neighbors in C.
2. $\mathcal{P}_{1,2}$: other cliques.



Want to show: with high probability,
each sparse vertex is safe and each clique has two safe vertices.

Bad events for the Lovász Local Lemma.

1. For each $v \in S$, $\mathcal{S}_v$ is the event that v is not safe;
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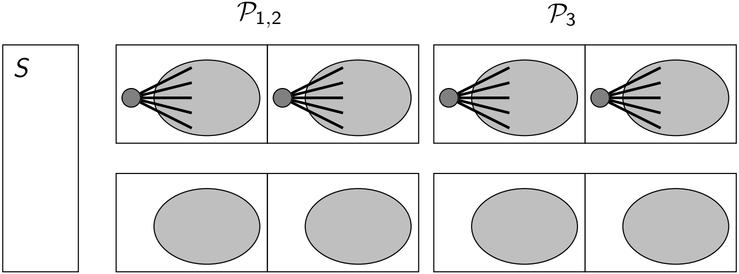
Assume none of the bad events happen. Then how to finish the coloring.



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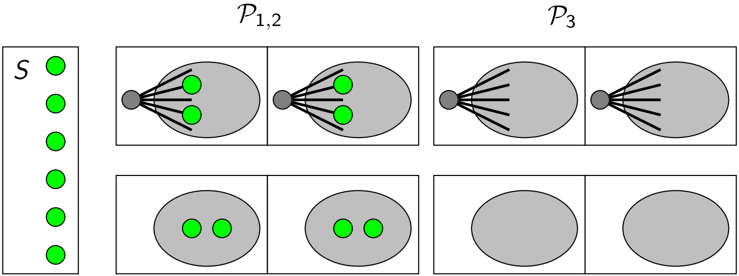

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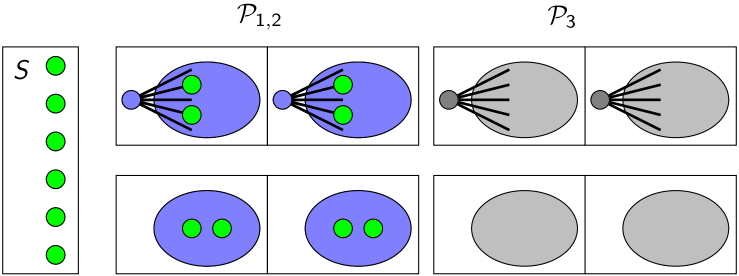

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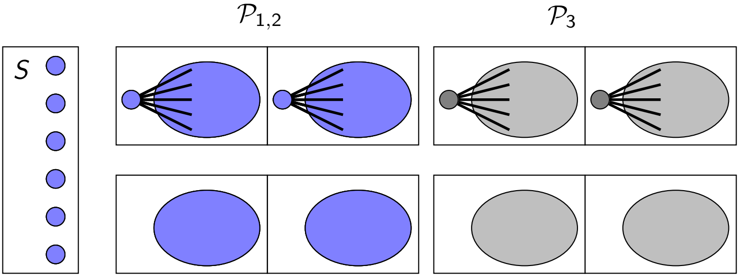

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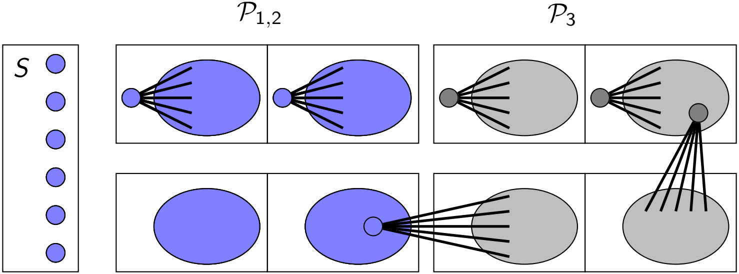

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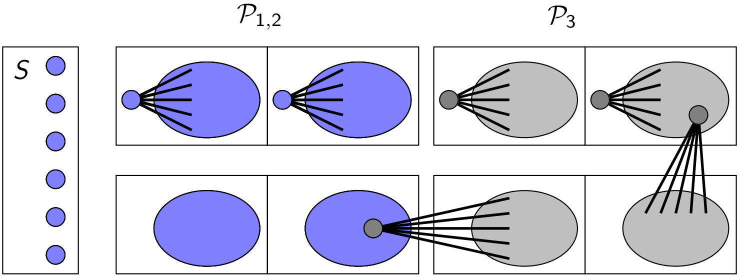

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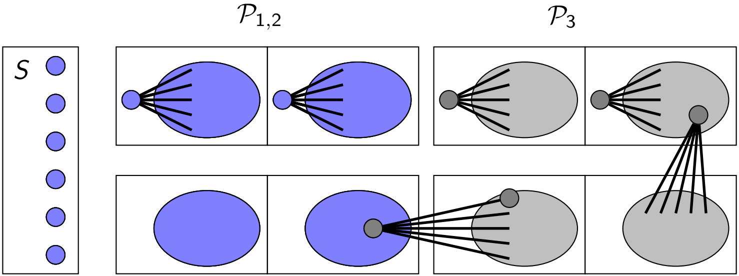

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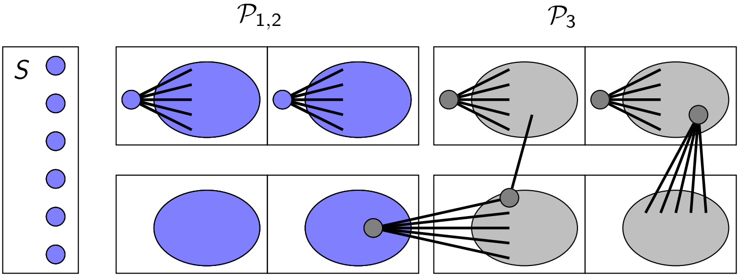

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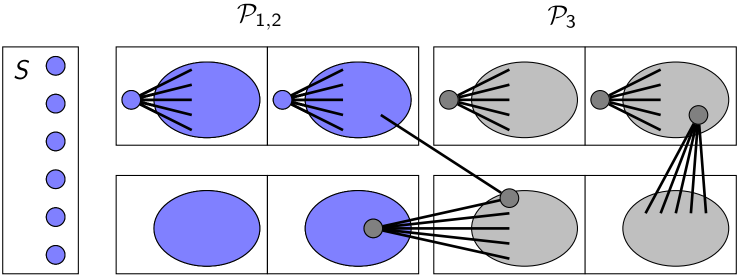

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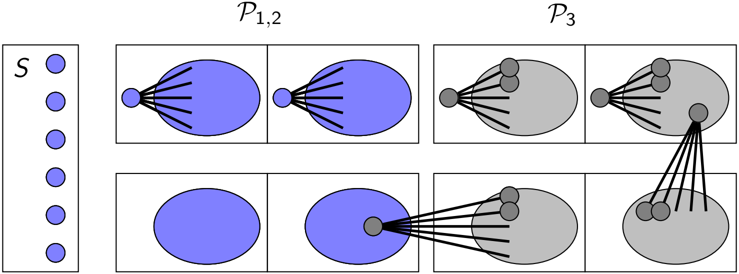

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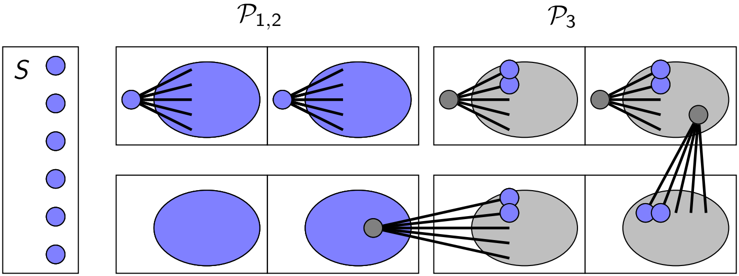

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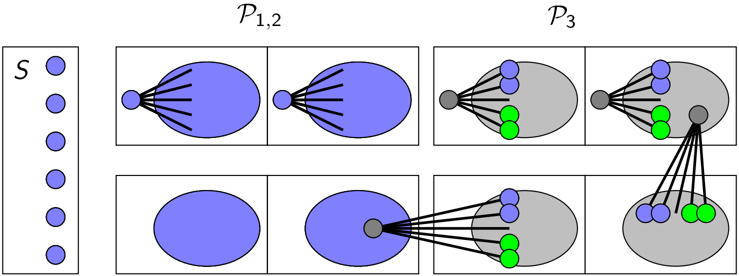

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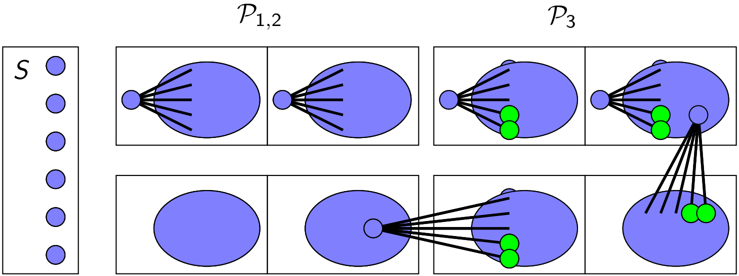

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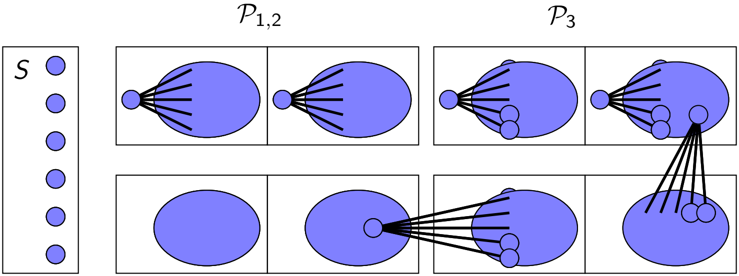

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Consider a set of (bad) events $\mathcal{E}$ where each E satisfies the following:

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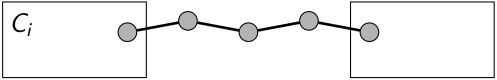
One event is not mutually independent to at most Δ^5 other events.



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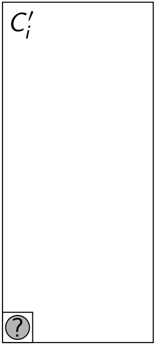

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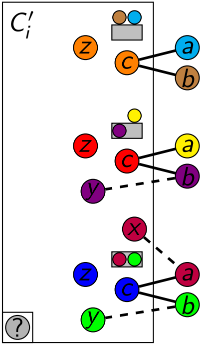

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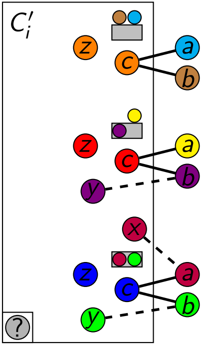

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Second, show that X is concentrated around its mean.

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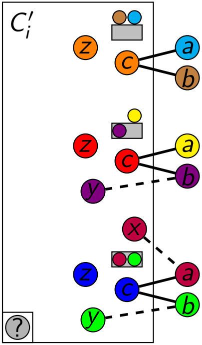

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Azuma's Inequality!

- For each $C_i \in \mathcal{P}_{1,2}$, let $\mathcal{E}_{1,i}$ be the event that C_i does not contain two safe vertices; Let $|C_i| = \Delta - 2$ so that $C_i \in \mathcal{P}_{1,2}$. At least 0.1314Δ P_3's.

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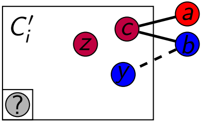

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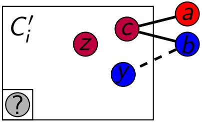

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$z \in C_i' - T_i$

$\alpha \in L(a) - L(c)$
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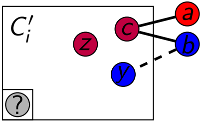

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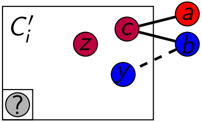
Let $A_{y,z}^{\alpha,\beta,\gamma}$ be the event that the following holds:

1. α is used on a and none of the rest of $N(a) \cup T_i$;
2. β is used on b and y and none of the rest of $N(b) \cup N(y) \cup T_i$;
3. γ is used on c and z and none of the rest of T_i.

$$Pr(A_{y,z}^{\alpha,\beta,\gamma}) \geq (\Delta - 1)^{-5} \left(1 - \frac{1}{\Delta-1}\right)^{|T_i \cup N(a)| + |T_i \cup N(y) \cup N(b)| + |T_i|} \geq \Delta^{-5} e^{-3.4}$$

There are $\frac{\Delta}{3} \frac{6\Delta}{7} \frac{2\Delta}{3} \frac{6\Delta}{7} \Delta$ choices for indices. Thus, the probability that $A_{y,z}^{\alpha,\beta,\gamma}$ happens for some choice of indices is at least 0.00375.

- For each $C_i \in \mathcal{P}_{1,2}$, let $\mathcal{E}_{1,i}$ be the event that C_i does not contain two safe vertices; Let $|C_i| = \Delta - 2$ so that $C_i \in \mathcal{P}_{1,2}$. At least 0.1314Δ P_3's.



Assume $L(c) \cap L(a) < \frac{2}{3}\Delta$ and $L(c) \cap L(b) \geq \frac{2}{3}\Delta$.

$y \in C_i' - T_i - N(b)$
$z \in C_i' - T_i$

$\alpha \in L(a) - L(c)$
$\beta \in L(b) \cap L(y)$
$\gamma \in L(c) \cap L(z)$

Let $A_{y,z}^{\alpha,\beta,\gamma}$ be the event that the following holds:

1. α is used on a and none of the rest of $N(a) \cup T_i$;
2. β is used on b and y and none of the rest of $N(b) \cup N(y) \cup T_i$;
3. γ is used on c and z and none of the rest of T_i.

$$Pr(A_{y,z}^{\alpha,\beta,\gamma}) \geq (\Delta - 1)^{-5} \left(1 - \frac{1}{\Delta - 1}\right)^{|T_i \cup N(a)| + |T_i \cup N(y) \cup N(b)| + |T_i|} \geq \Delta^{-5} e^{-3.4}$$

There are $\frac{\Delta}{3} \frac{6\Delta}{7} \frac{2\Delta}{3} \frac{6\Delta}{7} \Delta$ choices for indices. Thus, the probability that $A_{y,z}^{\alpha,\beta,\gamma}$ happens for some choice of indices is at least 0.00375.

Therefore, expected number of safe vertices is $2.5 \cdot 10^{-4}\Delta$.

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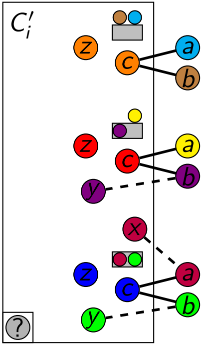
Theorem (Azuma's Inequality)

Let X be a random variable determined by n trials $T_1, \ldots, T_n$. For each i and any two possible outcomes, if the following holds:
$|E(X|T_1 = t_1, \ldots, T_i = t_i) - E(X|T_1 = t_1, \ldots, T_i = t_i')| \leq c_i$,
then $Pr(|X - E(X)| > t) \leq 2e^{-t^2/(2\sum c_i^2)}$.



If $v \in T_i \cup C_i'$, then $c_v \leq 2$.
Sum of c_v^2 is at most 8Δ.

For large Δ,

$$2e^{\frac{-(2.5\cdot 10^{-4}\Delta - 2)^2}{80\Delta}} \leq \Delta^{-6}$$

1. Decompose G "nicely" so we can analyze smaller pieces.

Theorem (Lovász Local Lemma)

Consider a set of (bad) events $\mathcal{E}$ where each E satisfies the following:

- $Pr(E) \leq p$
- *E is mutually independent to a set of all but at most d other events.*

If $ep(d+1) \leq 1$, then with positive probability, no event in $\mathcal{E}$ occurs.

2. Color randomly to obtain a good (enough) partial coloring.

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then $Pr(|X - E(X)| > t) \leq 2e^{-t^2/(2\sum c_i^2)}$.

3. For the remaining graph, color greedily to show that G cannot exist.

Thank you!