A Refinement of the Corrádi-Hajnal Theorem

Elyse Yeager
University of Illinois at Urbana-Champaign
Joint work with H. Kierstead and A. Kostochka

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Corrádi-Hajnal Theorem

Theorem 1

[Corradi, Hajnal 1963] Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\delta(H) \geq 2k$. Then H contains k vertex-disjoint cycles.

Corrádi-Hajnal Theorem

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Corollary 2

Let $n = 3k$, and let H be an n-vertex graph with $\delta(H) \geq 2k$. Then H contains k vertex-disjoint triangles.

Refinements

Theorem 3

[Aigner, Brandt 1993]: Let H be an n-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then H contains each 2-factor.

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Theorem 4
[Kostochka, Yu 2011]: Let $n \geq 3$ and H be an n-vertex graph with $\sigma_2(H) \geq 4n/3 - 1$. Then H contains each 2-factor.

Refinements

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Definition

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Theorem 4

[Kostochka, Yu 2011]: Let $n \geq 3$ and H be an n-vertex graph with $\sigma_2(H) \geq 4n/3 - 1$. Then H contains each 2-factor.

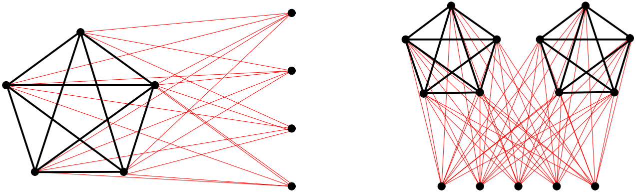
Theorem 5

[Fan, Kierstead 1996]: Let $n \geq 3$ and H be an n-vertex graph with $\delta(H) \geq \frac{2n-1}{3}$. Then H contains the square of the n-vertex path.

Refinements

Theorem (1)

Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\delta(H) \geq 2k$. Then H contains k vertex-disjoint cycles.



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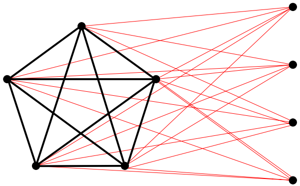
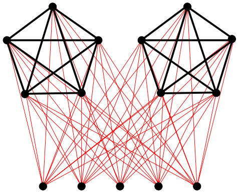




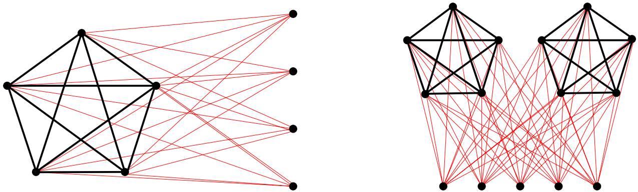
Theorem 6

[Enomoto 1998; Wang 1999]: Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\sigma_2(H) \geq 4k-1$. Then H contains k vertex-disjoint cycles.

Refinements

Theorem (1)

Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\delta(H) \geq 2k$. Then H contains k vertex-disjoint cycles.



Theorem 6

[Enomoto 1998; Wang 1999]: *Let $k \geq 1, n \geq 3k$, and let H be an n-vertex graph with $\sigma_2(H) \geq 4k - 1$. Then H contains k vertex-disjoint cycles.*

Theorem 7

[Kierstead, Kostochka, Y.]: *Let $k \geq 3$, $n \geq 3k + 1$, and let H be an n-vertex graph with $\sigma_2(H) \geq 4k - 2$ and $\alpha(H) \leq n - 2k$. Then H contains k vertex-disjoint cycles.*

Proof Sketch: Theorem 7

Theorem (7)

[Kierstead, Kostochka, Y.]: *Let $k \geq 3$, $n \geq 3k+1$, and let H be an n-vertex graph with $\sigma_2(H) \geq 4k-2$ and $\alpha(H) \leq n-2k$. Then H contains k vertex-disjoint cycles.*

Idea of Proof: Suppose G is an edge-maximal counterexample. Let $\mathcal{C}$ be a set of disjoint cycles in G such that:

- $|\mathcal{C}|$ is maximized,
- subject to the above, $\sum_{C \in \mathcal{C}} |C|$ is minimized, and
- subject to both other conditions, the length of a longest path in $G - \bigcup \mathcal{C}$ is maximized.

Proof of Therem 7

Goal (1)

$R := G - \mathcal{C}$ *is a path*

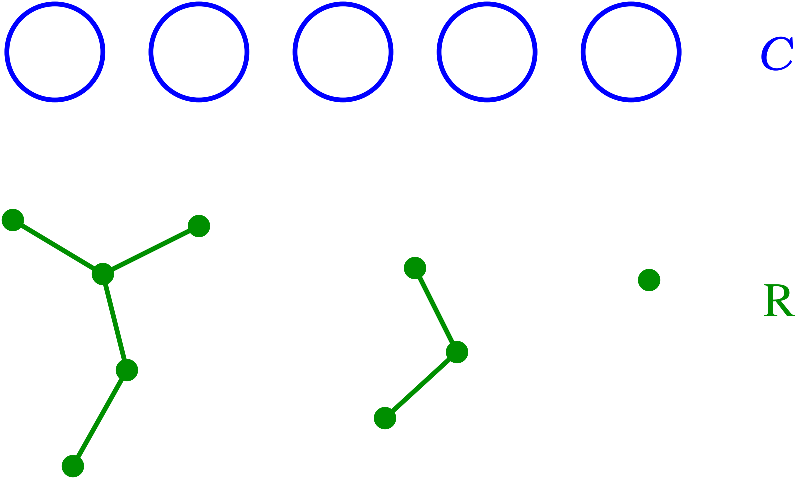
Goal (2)

$|R| = 3$

Goal (3)

$|R| \geq 4$

Goal 1



Notice R is a forest. If R is not a path, it has at least three buds. Let a be an endpoint of a longest path P, and let c be a bud not on P.

Goal 1: R is a Path

Claim 1

Suppose R is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

R is a subdivided star.

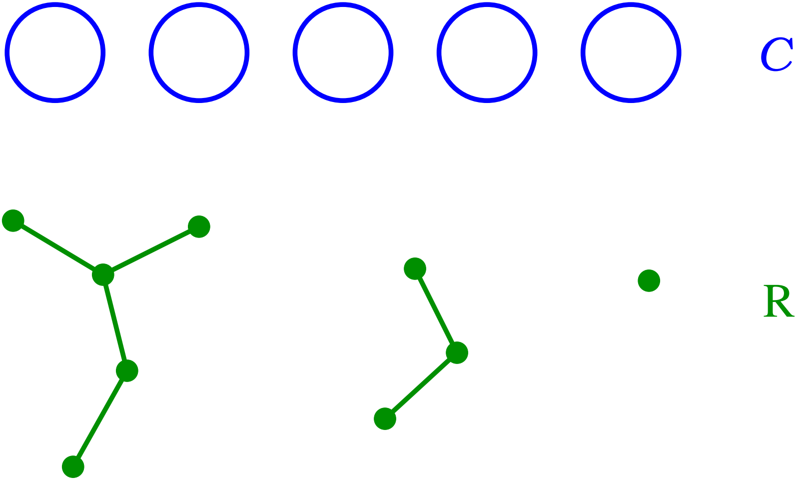
Claim 4

R is a path or a star.

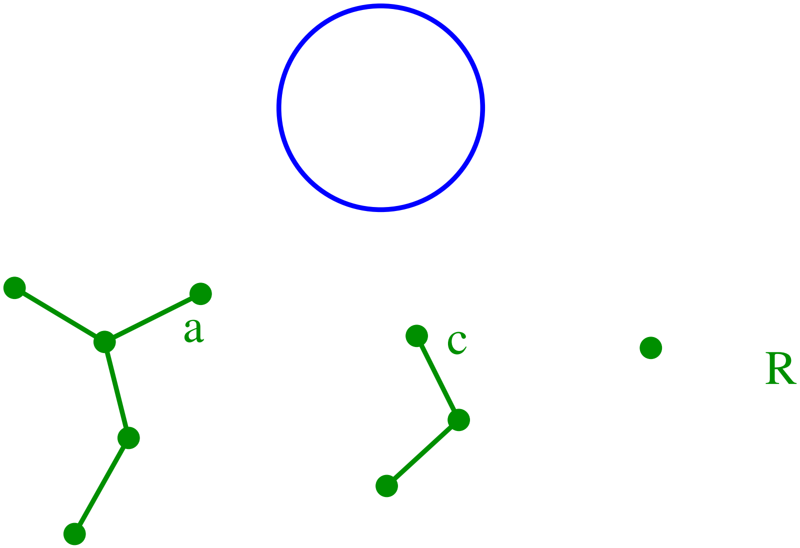
Claim 5

R is a path.

Claim 1



Claim 1



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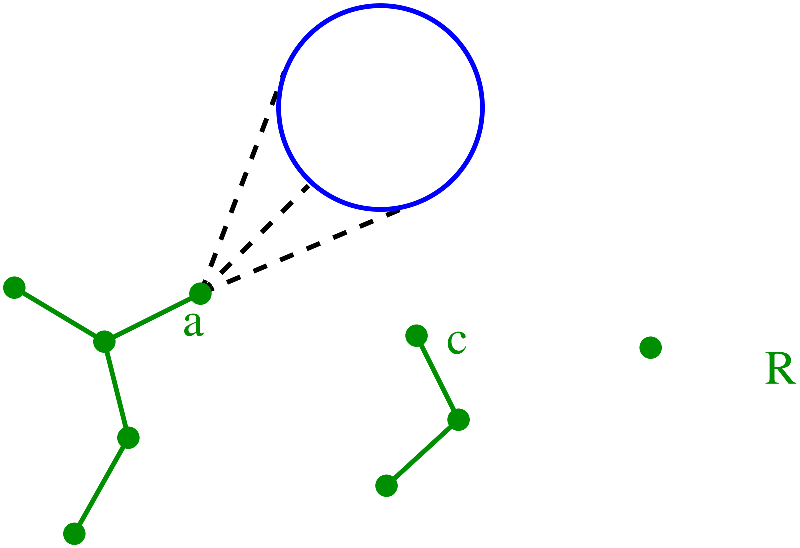

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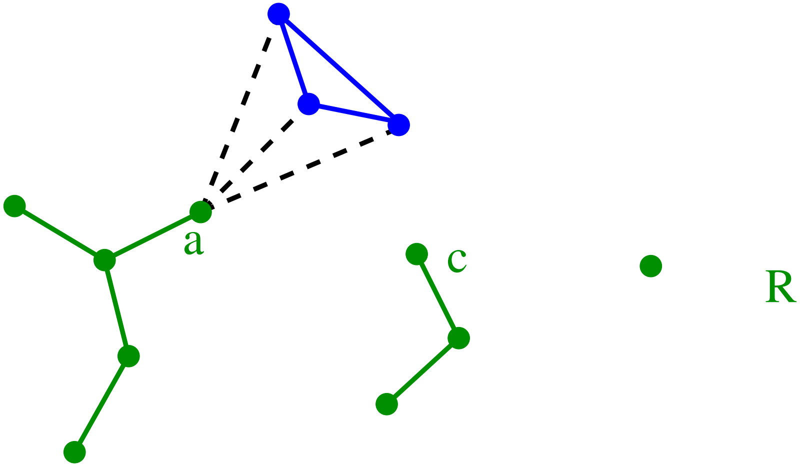

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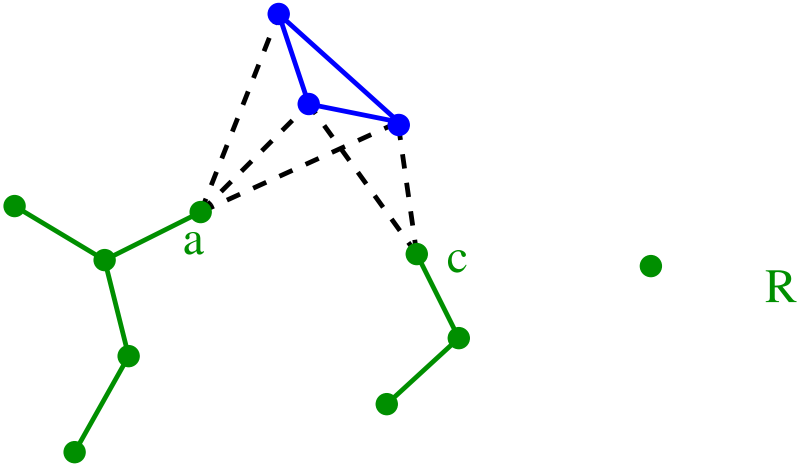

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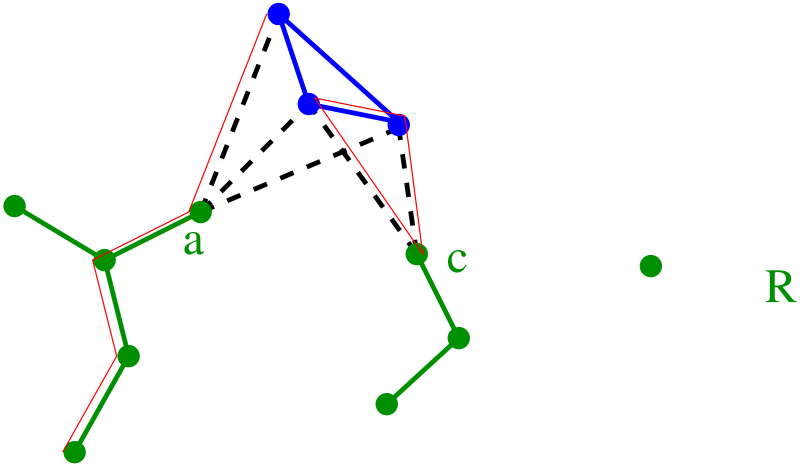

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So, $||\{a, c\}, C|| \leq 4$ for every $C \in \mathcal{C}$.

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We can now show $||\{a, c\}, C|| = 4$ by a counting argument, using the minimum degree sum of G–recall, a and c are nonadjacent. This proves Claim (1).

The same counting argument shows that a and c must have one neighbor in R, so R has no isolated vertices.

Goal 1: R is a Path

Claim 1

Suppose R is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

R is a subdivided star.

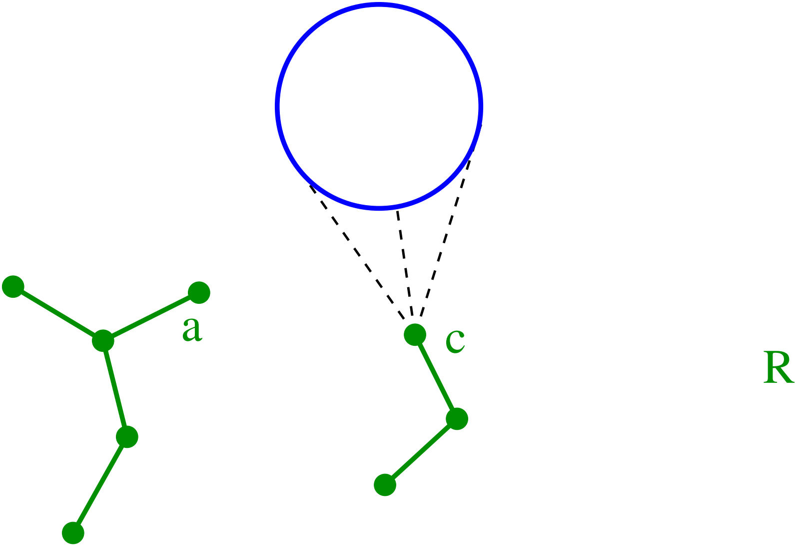
Claim 4

R is a path or a star.

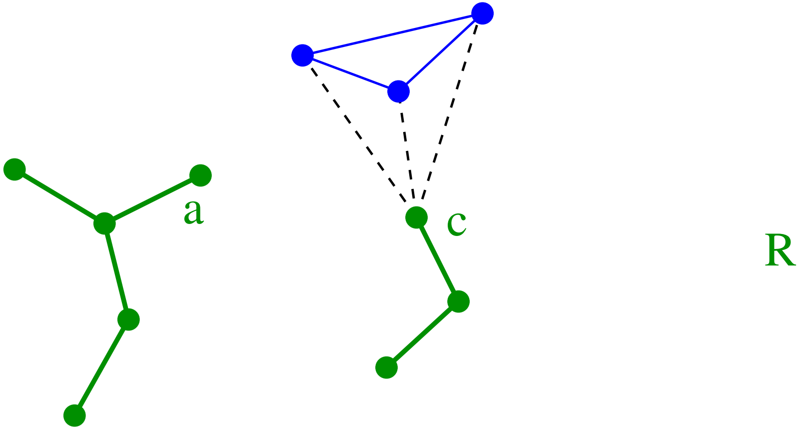
Claim 5

R is a path.

Claim 2



Claim 2



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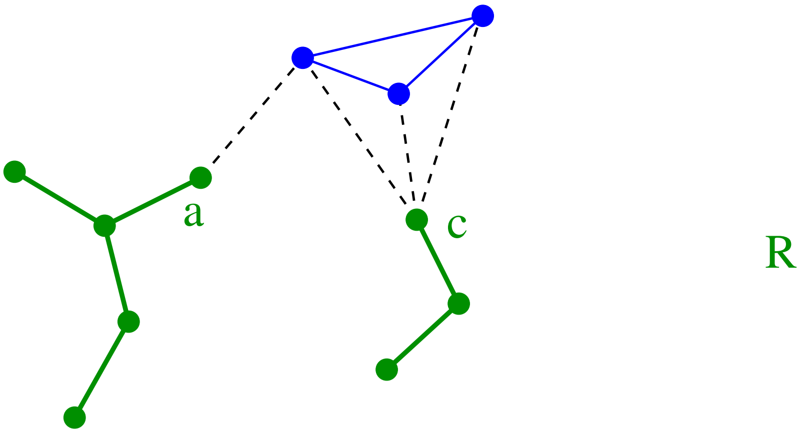

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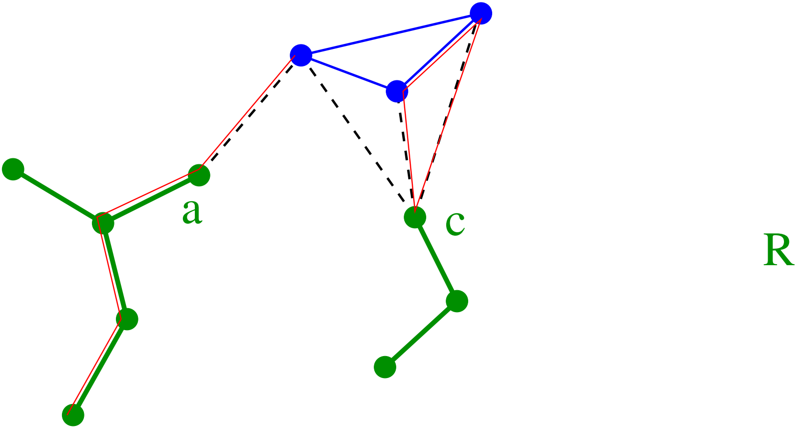
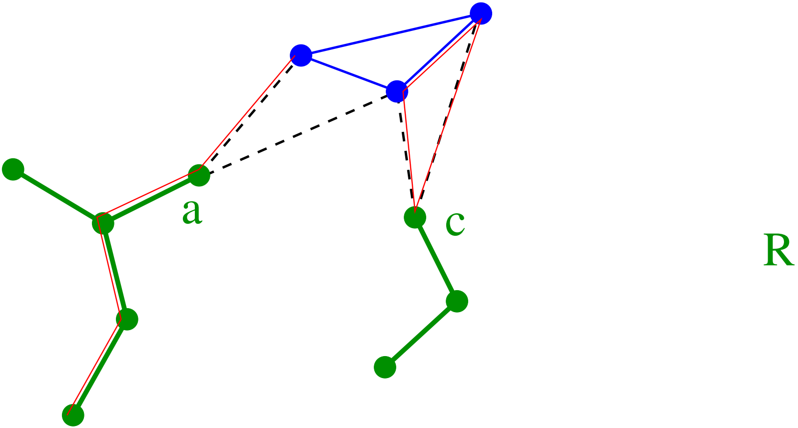


Claim 2

So we see that c can have at most 2 neighbors in any cycle $C \in \mathcal{C}$. By degree-sum considerations, c must have precisely two neighbors in each cycle $C \in \mathcal{C}$. This tells us that a, as well, has precisely 2 neighbors to every cycle $C \in \mathcal{C}$.

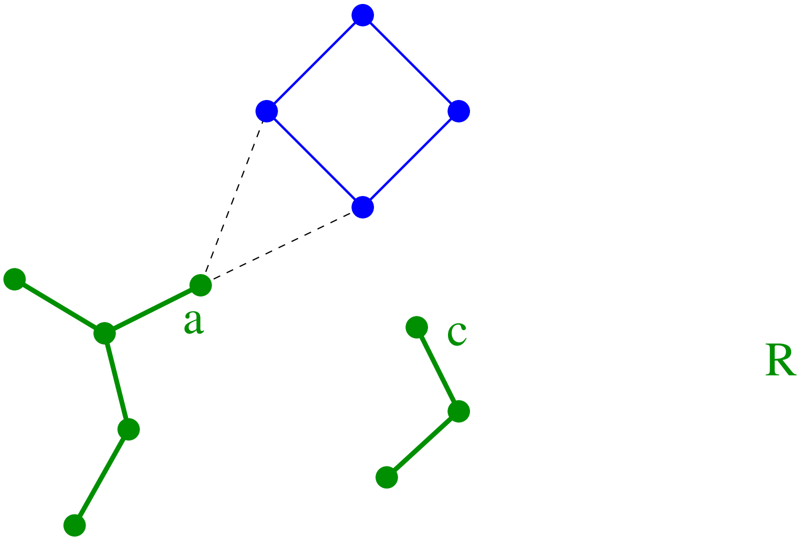
It remains only to show that no two leaves in R have different sets of neighbors, and if $|C| = 4$, the neighbors of our leaves are nonadjacent.

Claim 2



So if $|C| = 3$, then $N(a) \cap C = N(c) \cap C$, as desired.

Claim 2



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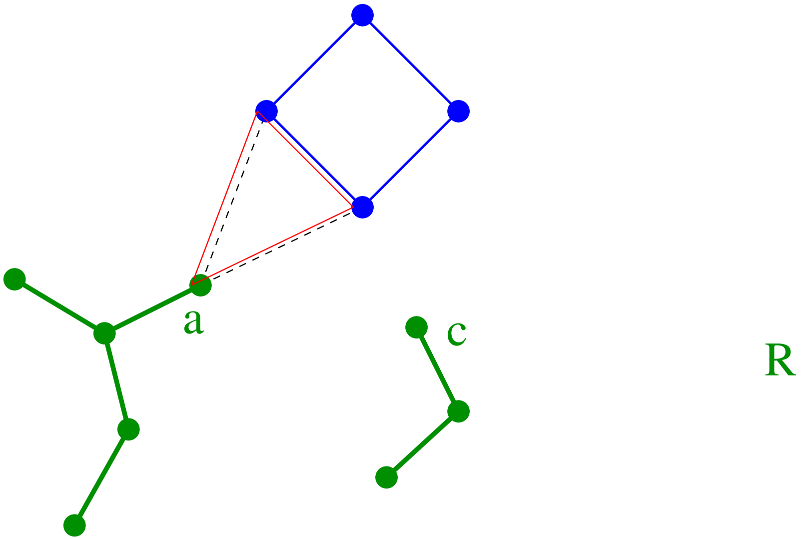

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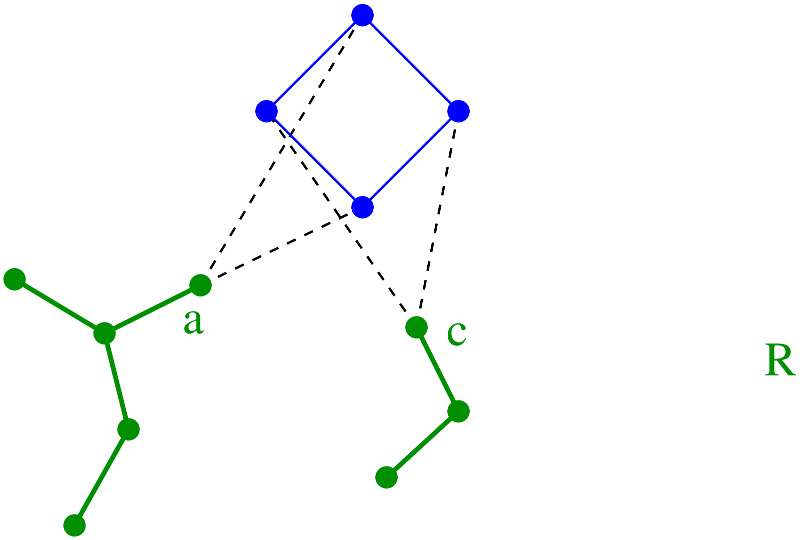

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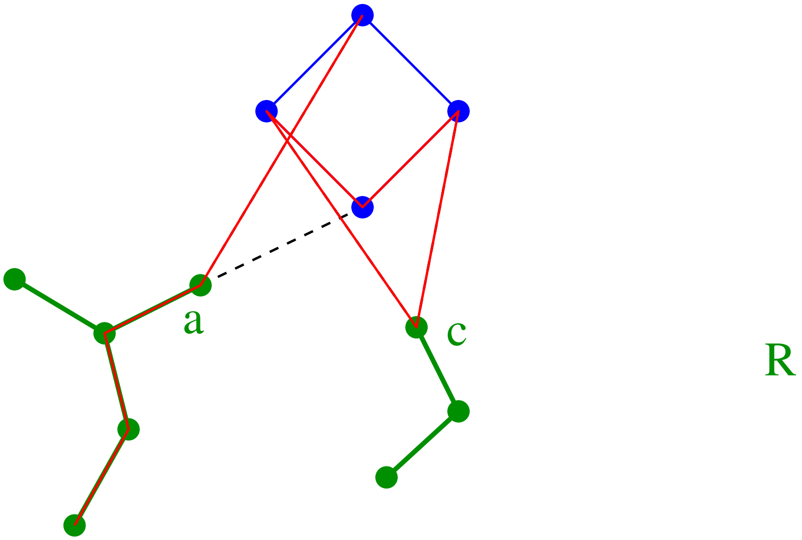

Claim 2

This proves Claim 2.

Goal 1: R is a Path

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Suppose R is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

R is a subdivided star.

Claim 4

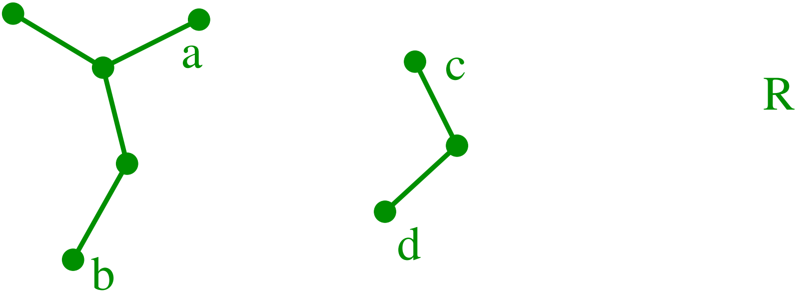
R is a path or a star.

Claim 5

R is a path.

Claim 3

Suppose R is not a subdivided star. Then it has four leaves a, b, c, d such that the paths aRb and cRd exist and are disjoint.



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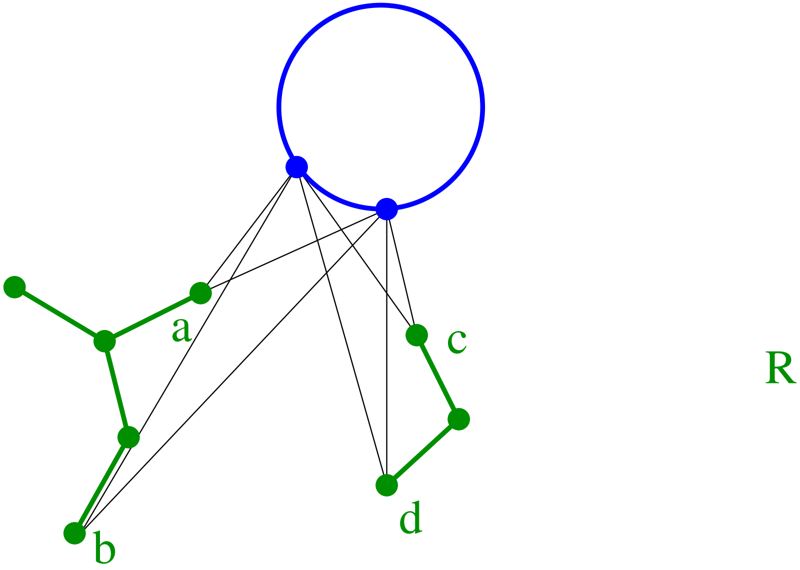

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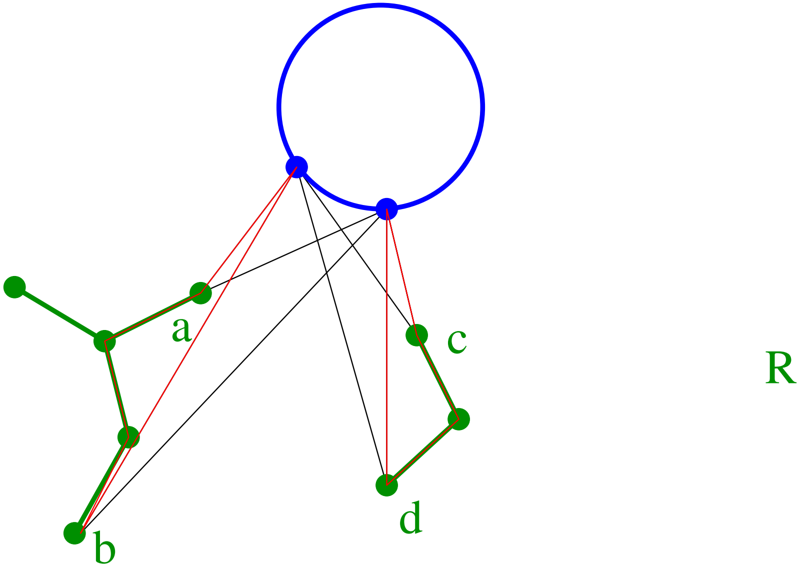

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Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

R is a subdivided star.

Claim 4

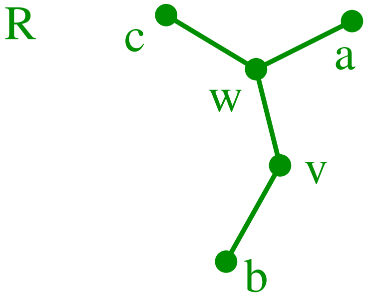
R is a path or a star.

Claim 5

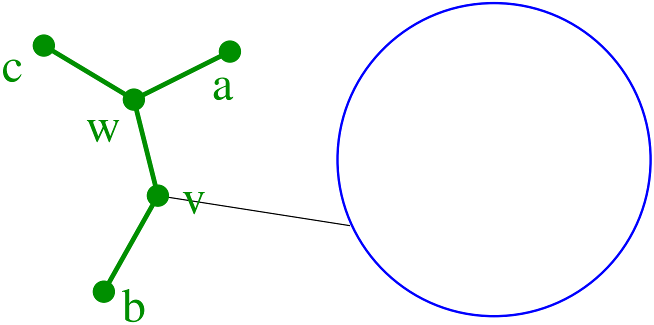
R is a path.

Claim 4

Suppose R is not a path or a star. We know it is a subdivided star, so there must be some unique vertex w with degree at least three. Since we assume it is not a star, there is also a vertex v of degree 2. Further, there exist leaves a, b, c so that vRb does not contain w and is disjoint rom aRc.



Claim 4



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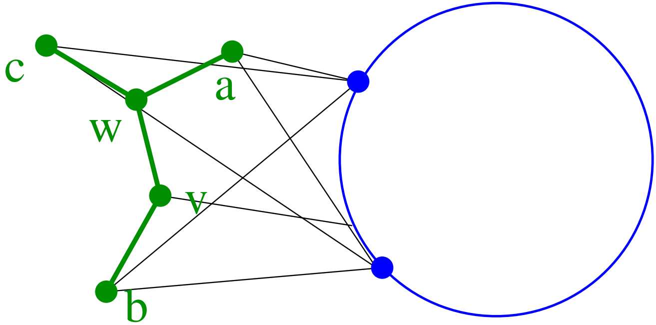

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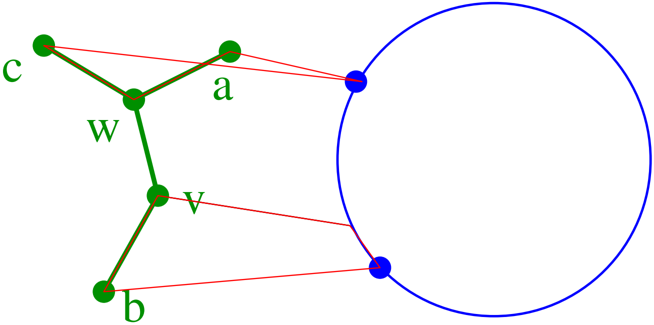

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Claim 3

R is a subdivided star.

Claim 4

R is a path or a star.

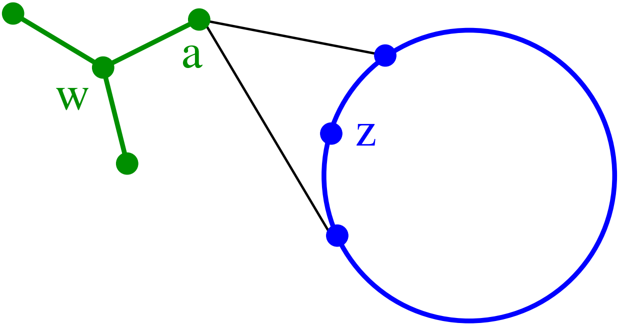
Claim 5

R is a path.

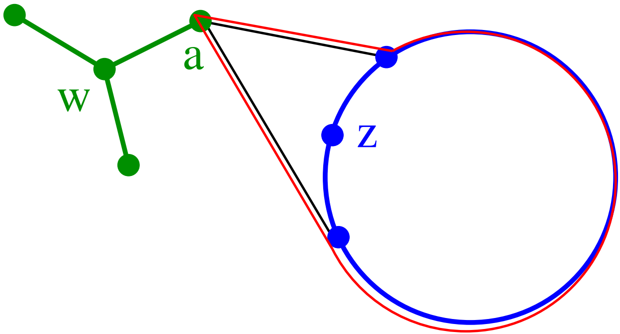
Claim 5

Suppose R is not a path. R has precisely one vertex w of degree at least 3.

Let z be an arbitrary vertex in $\mathcal{C} - N(a)$.



Claim 5



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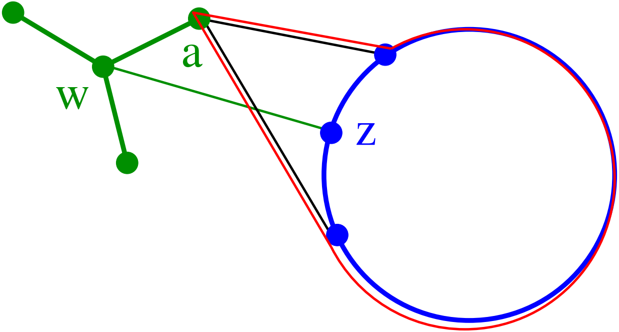

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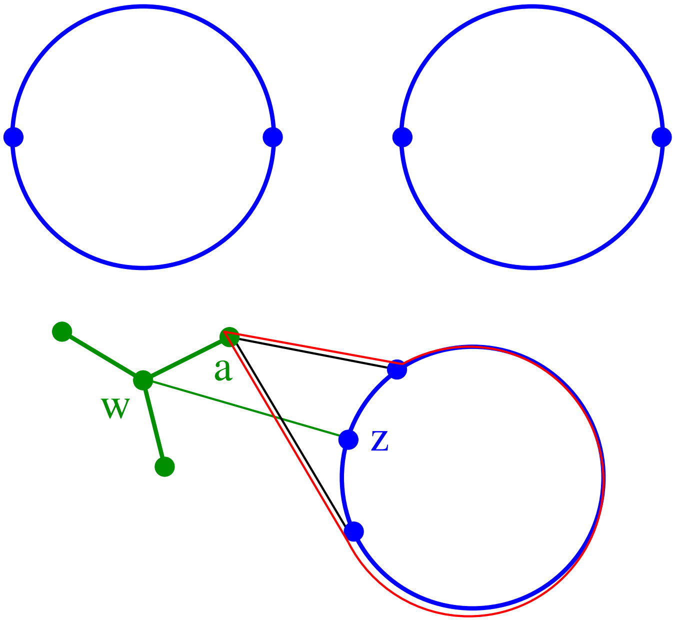

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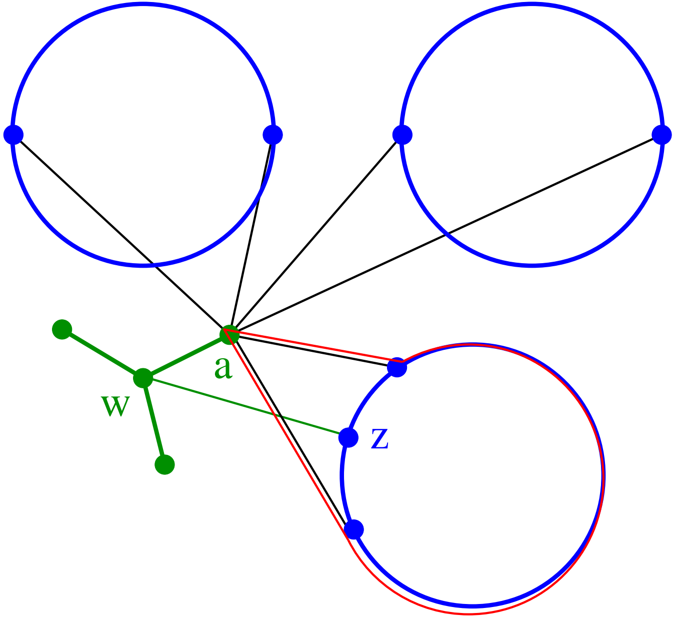

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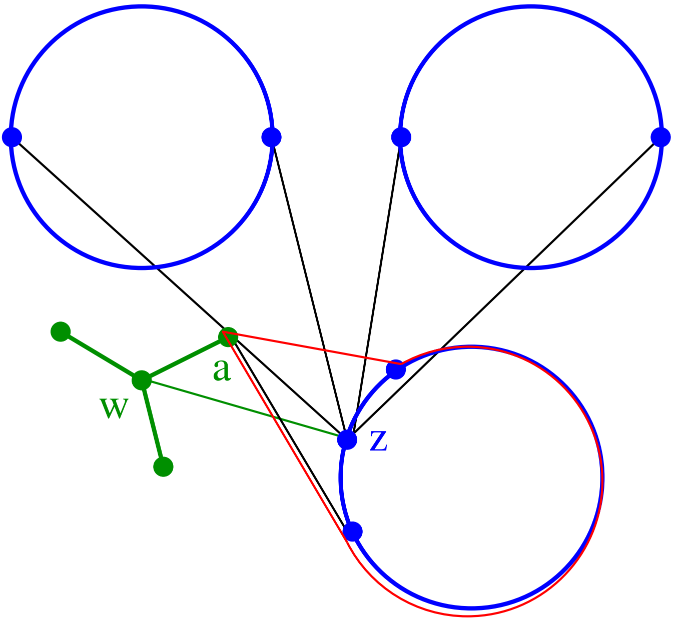

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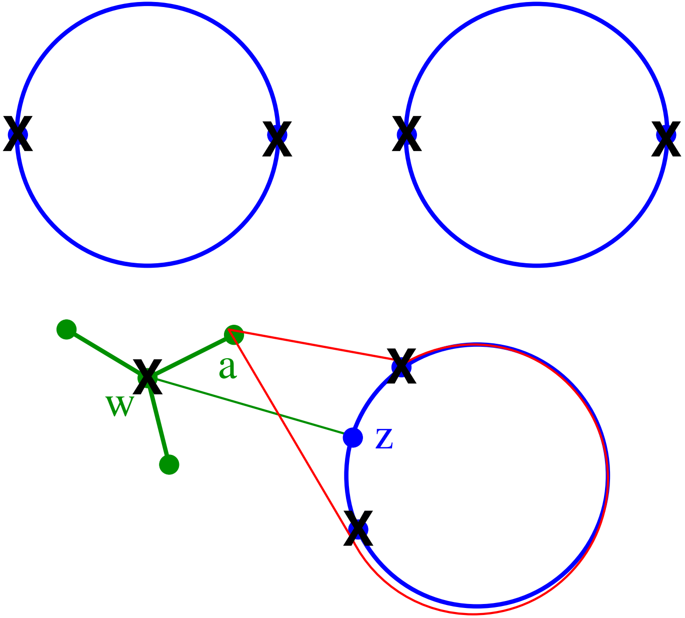

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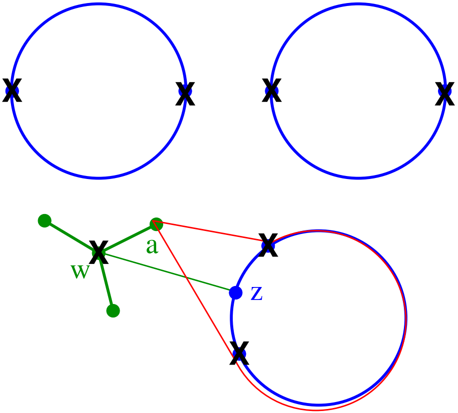


The independent set has size:

$$|V(G)| - 2(k-1) - 1 = n - 2k + 1$$

but we assumed $\alpha(G) \leq n - 2k$, a contradiction. This proves Claim 5, also Goal 1, that R is a path.

Goal 1: R is a Path

Claim 1

Suppose R is not a path. $||\{a, c\}, C|| = 4$ for every $C \in \mathcal{C}$.

Claim 2

Suppose R is not a path. Then for all cycles $C \in \mathcal{C}$ and for all leaves c in R, a and c share exactly the same two neighbors in C. If $|C| = 4$, then those neighbors are nonadjacent.

Claim 3

R is a subdivided star.

Claim 4

R is a path or a star.

Claim 5

R is a path.

Proof of Therem 7

Goal (1)

$R := G - \mathcal{C}$ *is a path*

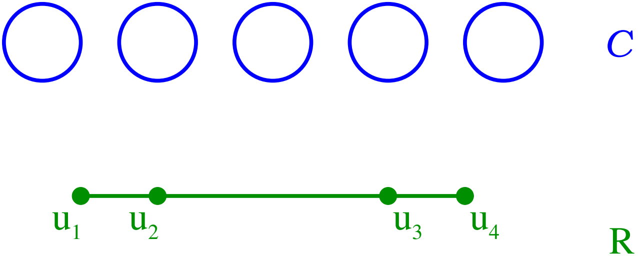
Goal (2)

$|R| = 3$

Goal (3)

$|R| \geq 4$

Goal 2: $|R| = 3$



We assume $|R| \geq 4$, and label the outermost four vertices of R as $F = \{u_1, u_2, u_3, u_4\}$.

Goal 2: $|R| = 3$

Claim 6

If $\|C, F\| \geq 7$ for any $C \in \mathcal{C}$, then

- $|C| = 3$
- $\|C, F\| = 7$
- *u_1 is adjacent to precisely x_1, x_2 in C, u_2 is adjacent to all three vertices of C, and x_1, x_2 each have precisely one neighbor in $\{u_3, u_4\}$ (or mirror case)*

Claim 7

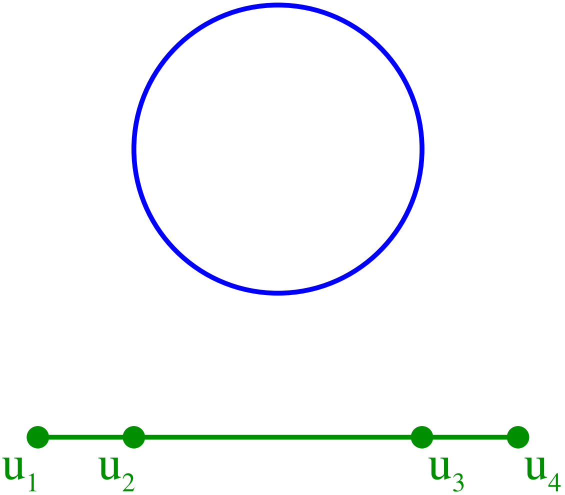
$k = 3$ and $\|C, F\| = 7$ for both $C \in \mathcal{C}$

Claim 8

$|R| = 3$

Claim 6

We suppose $\|C, F\| \geq 7$ for some $C \in \mathcal{C}$.



Claim 6

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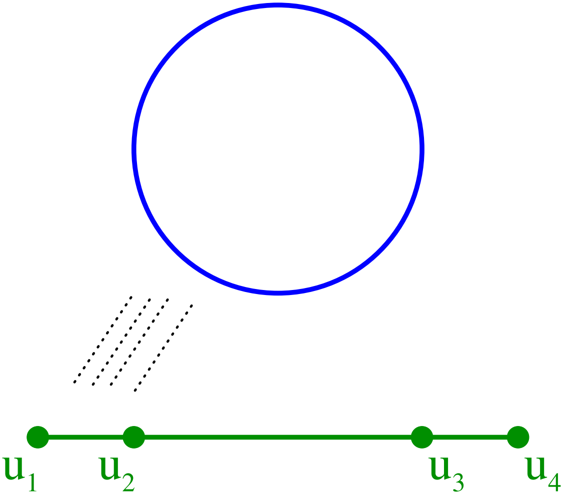

Claim 6

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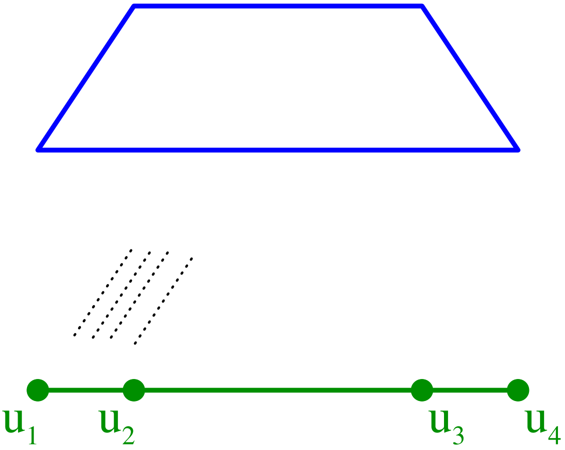

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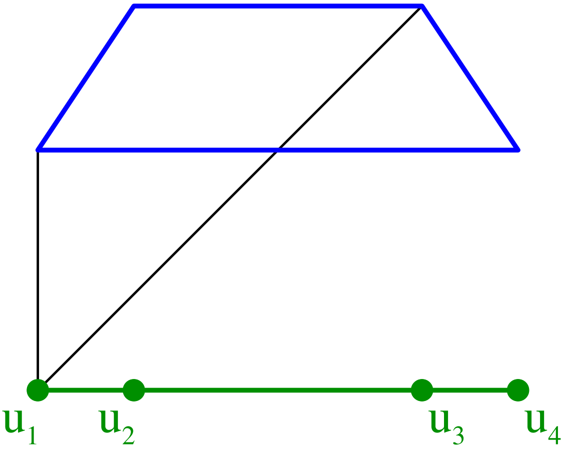

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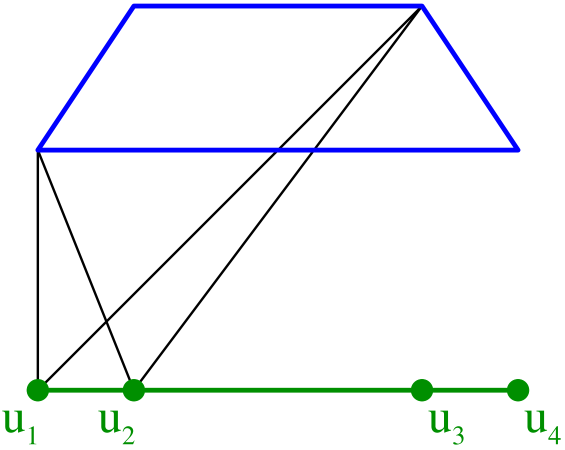

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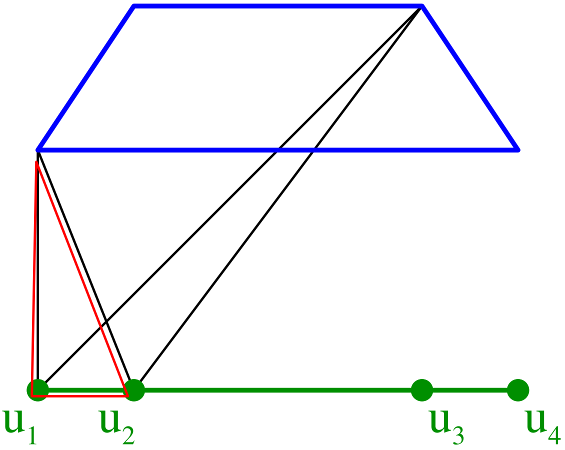

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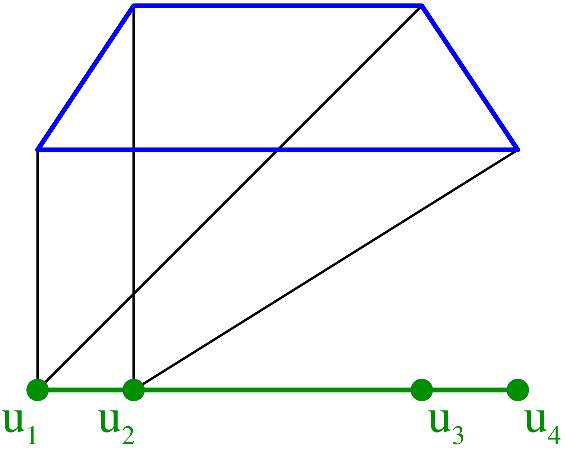

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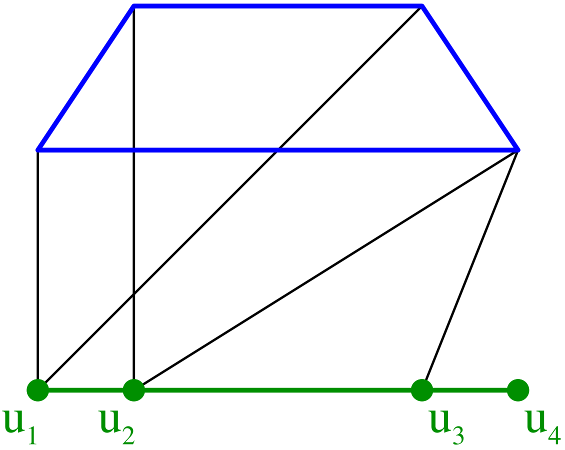

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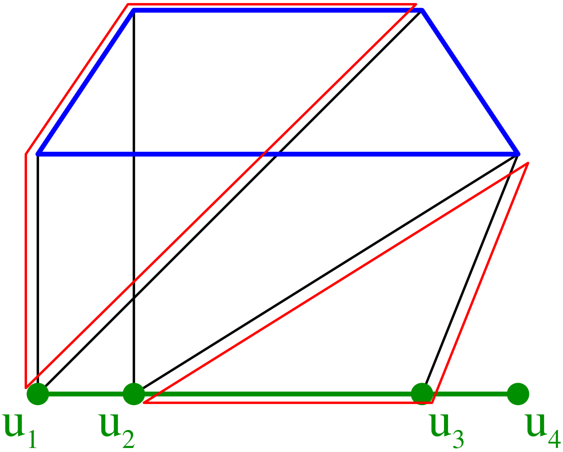

Goal 2: $|R| = 3$

Claim 6

If $\|C, F\| \geq 7$ *for any* $C \in \mathcal{C}$*, then*

- $|C| = 3$
- $\|C, F\| = 7$
- u_1 *is adjacent to precisely* x_1, x_2 *in* C*,* u_2 *is adjacent to all three vertices of* C*, and* x_1, x_2 *each have precisely one neighbor in* $\{u_3, u_4\}$ *(or mirror case)*

Claim 7

$k = 3$ *and* $\|C, F\| = 7$ *for both* $C \in \mathcal{C}$

Claim 8

$|R| = 3$

Claim 6

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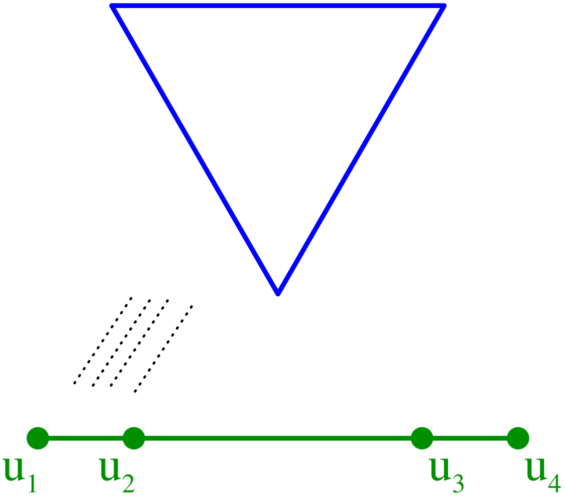

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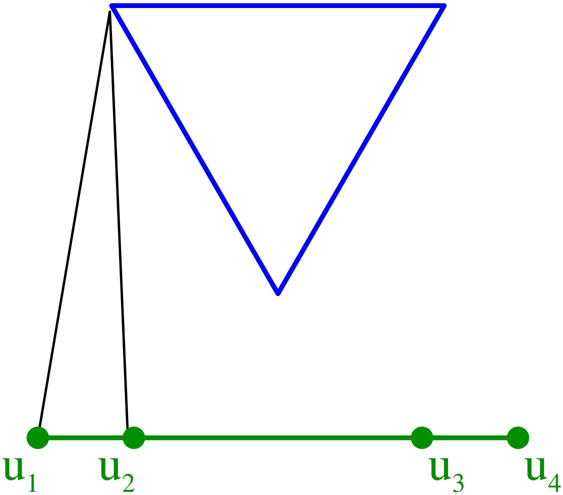

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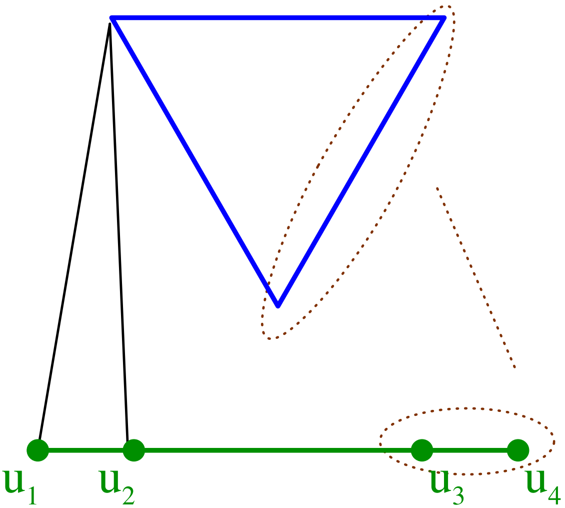

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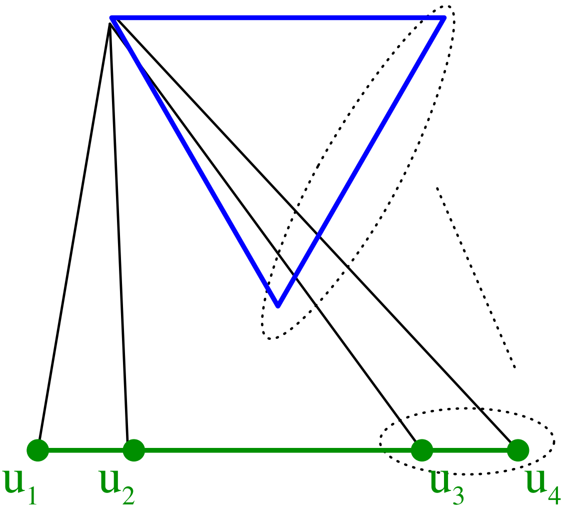
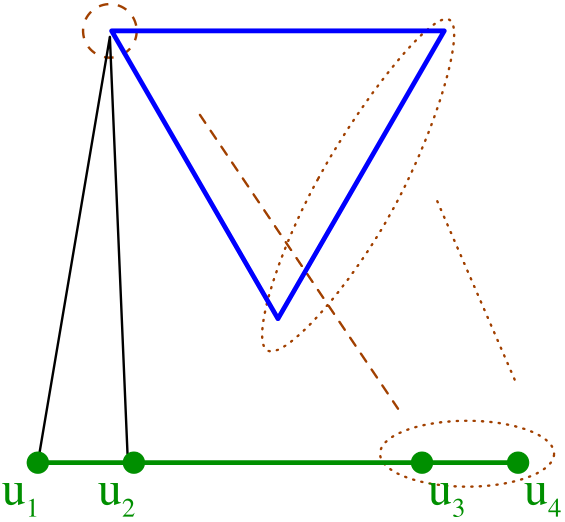


Claim 6

We suppose $\|C, F\| \geq 7$ for some $C \in \mathcal{C}$. $\|\{u_1, u_2, C\}\| \geq 5$.



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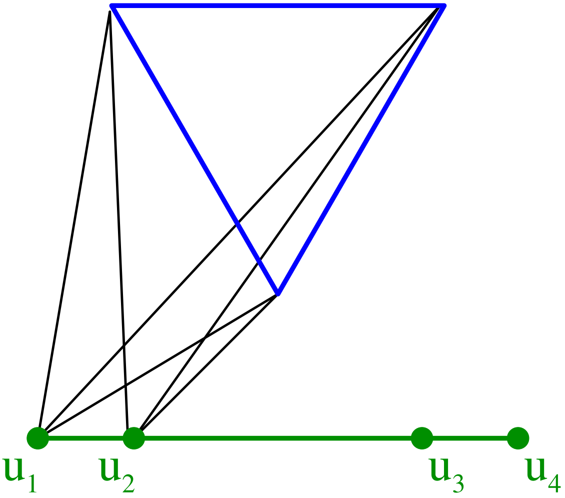

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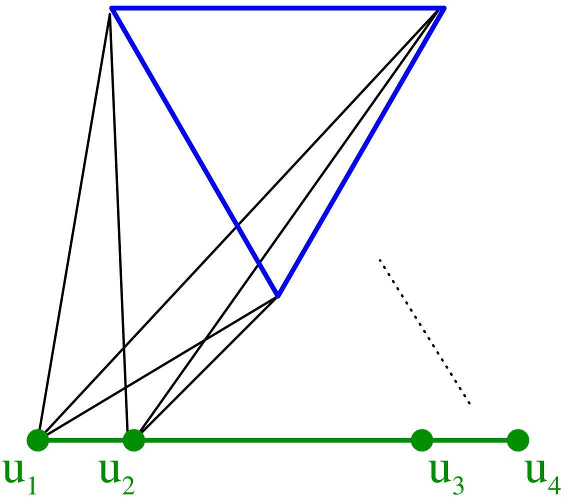

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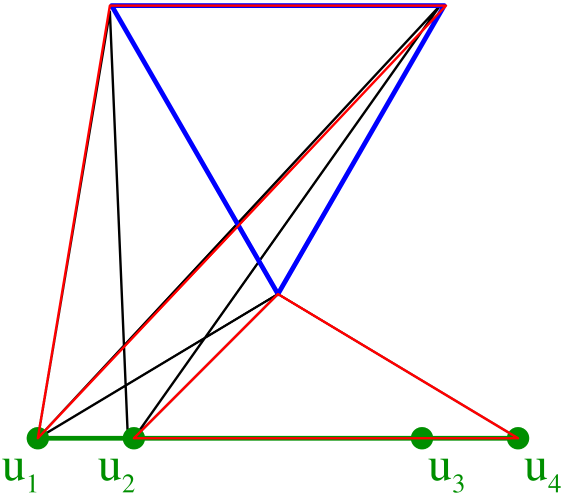

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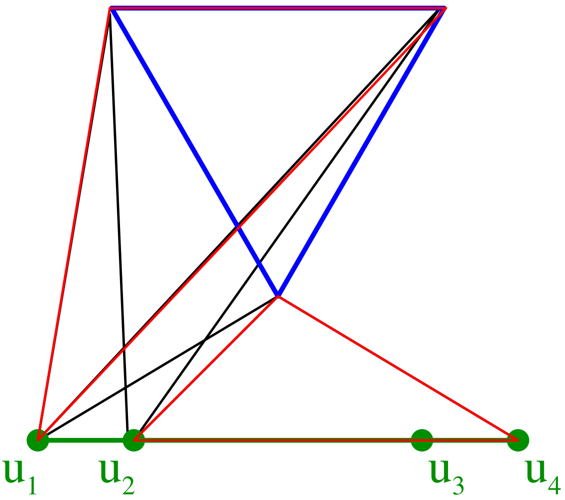

Goal 2: $|R| = 3$

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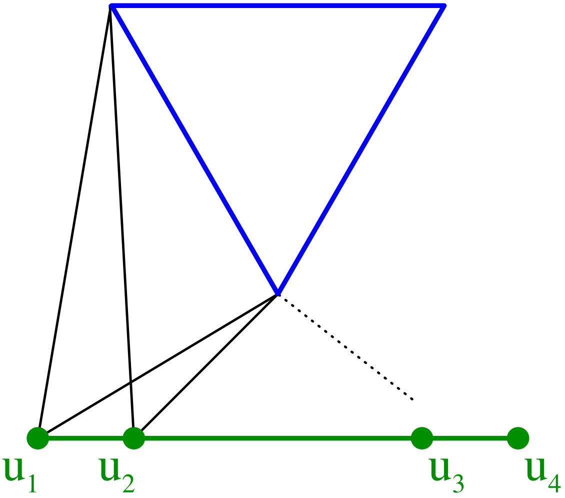

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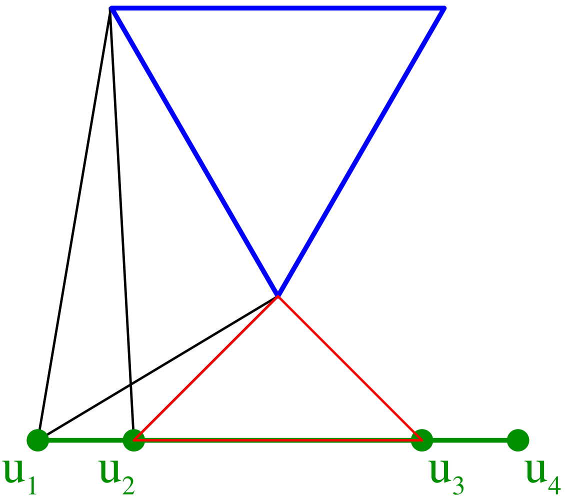

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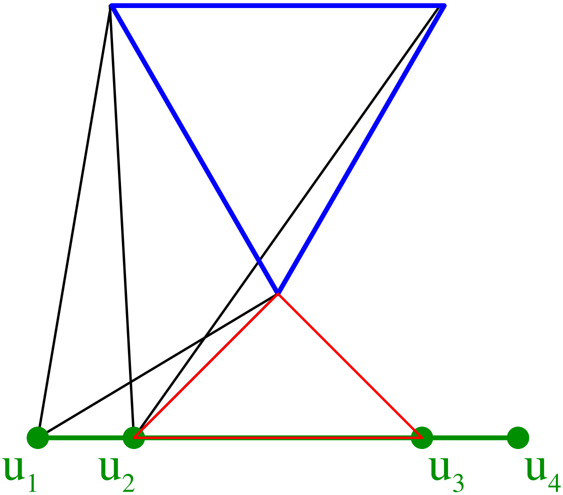

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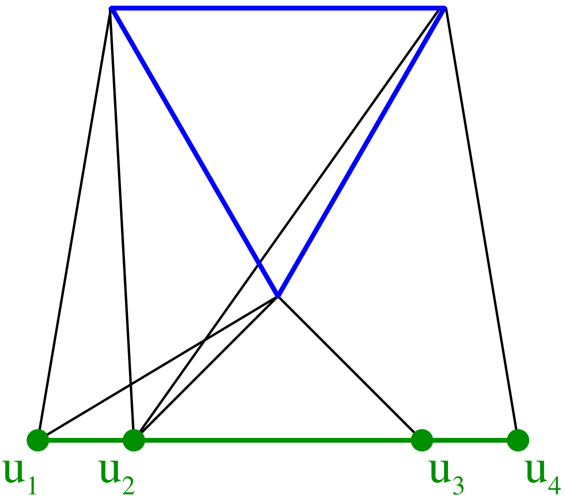

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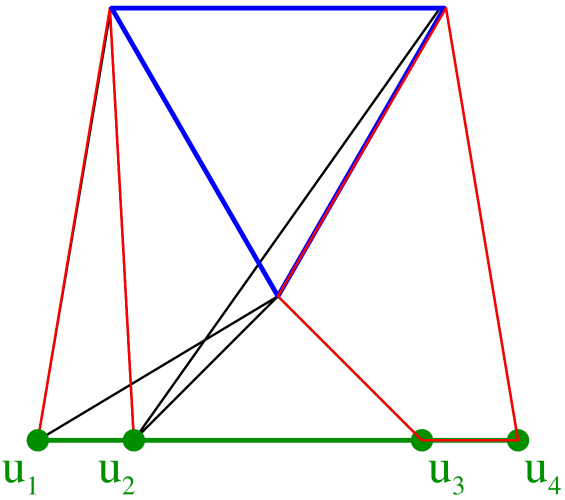

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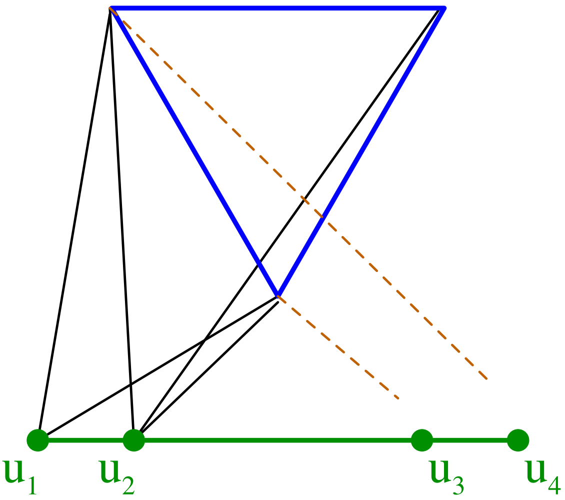

Goal 2: $|R| = 3$

Claim 6

If $\|C, F\| \geq 7$ *for any* $C \in \mathcal{C}$*, then*

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Claim 7

$k = 3$ *and* $\|C, F\| = 7$ *for both* $C \in \mathcal{C}$

Claim 8

$|R| = 3$

Claim 7

$$[d(u_1) + d(u_3)] + [d(u_2) + d(u_4)] \geq 2(4k - 2) = 8k - 4$$

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$$d(u_1) + d(u_3) + d(u_2) + d(u_4) = 6 + \|F, \mathcal{C}\| \leq 6 + 7(k-1) = 7k - 1$$

Claim 7

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$$d(u_1) + d(u_3) + d(u_2) + d(u_4) = 6 + \|F, \mathcal{C}\| \leq 6 + 7(k-1) = 7k - 1$$

Then $8k - 4 \geq 7k + 1$, and so $k \leq 3$.
We conclude $k = 3$ and $\|C, F\| = 7$ for both $C \in \mathcal{C}$, as desired.

Goal 2: $|R| = 3$

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If $\|C, F\| \geq 7$ for any $C \in \mathcal{C}$, then

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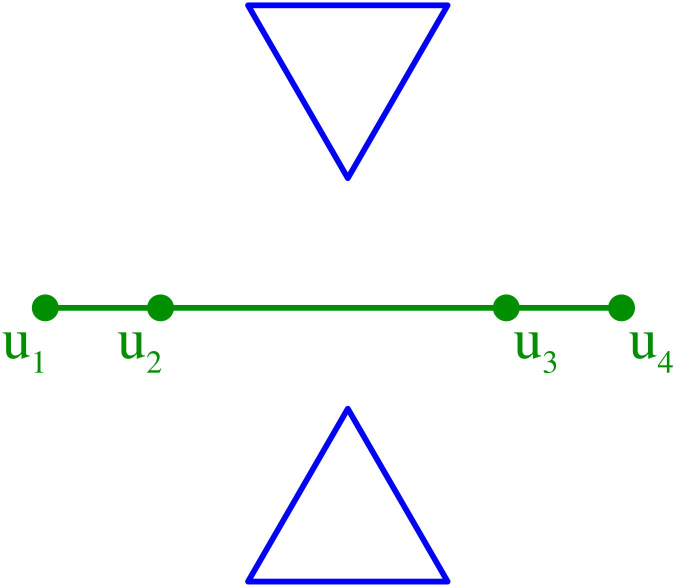

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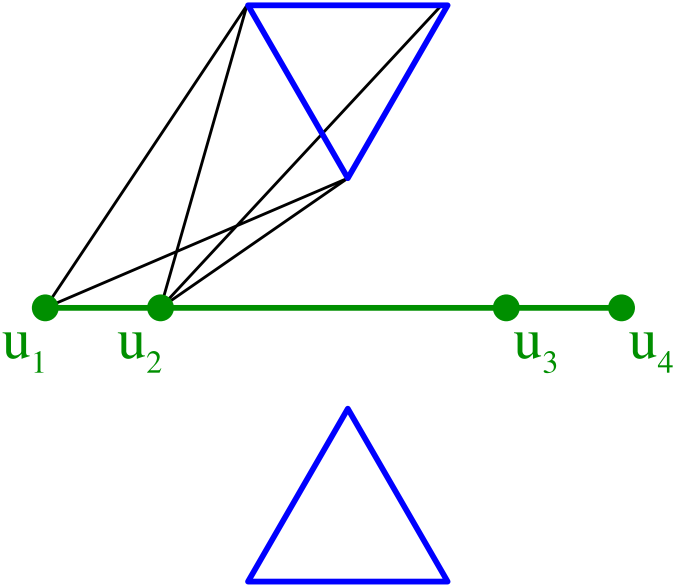

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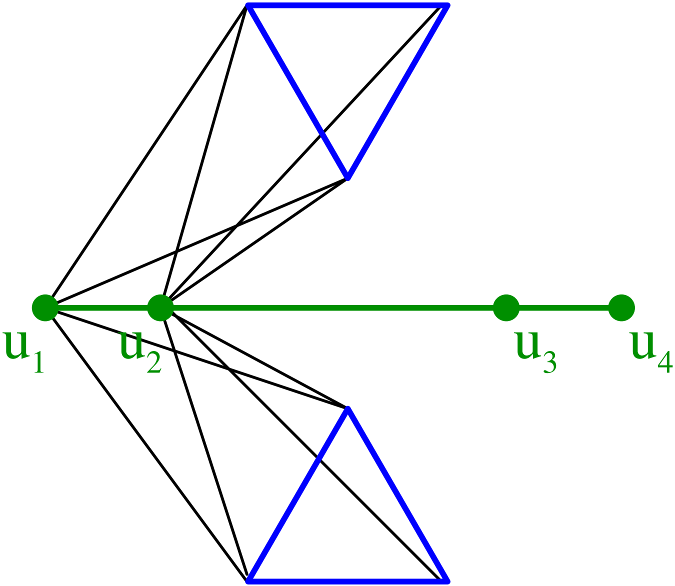

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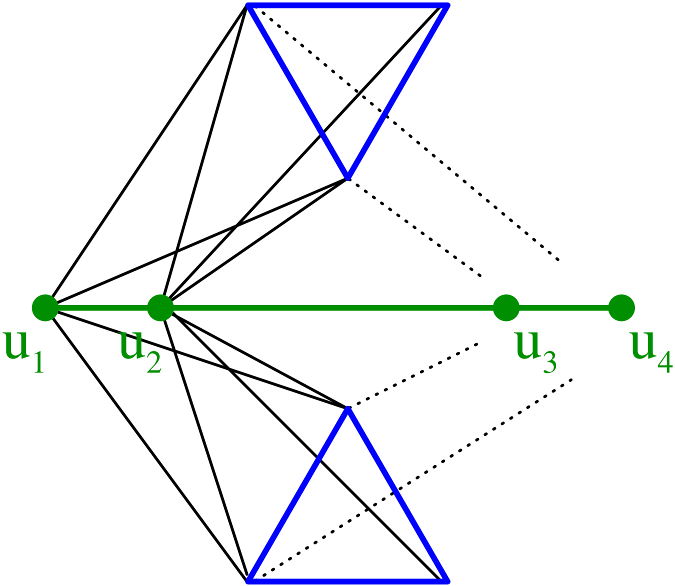

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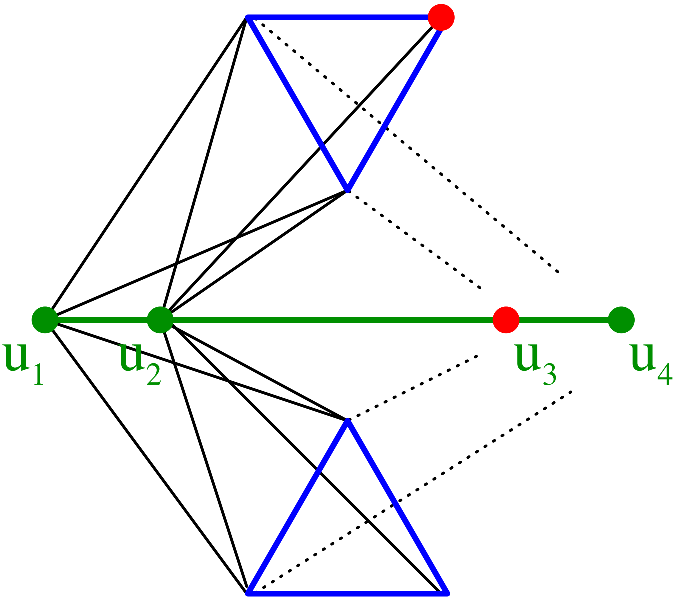
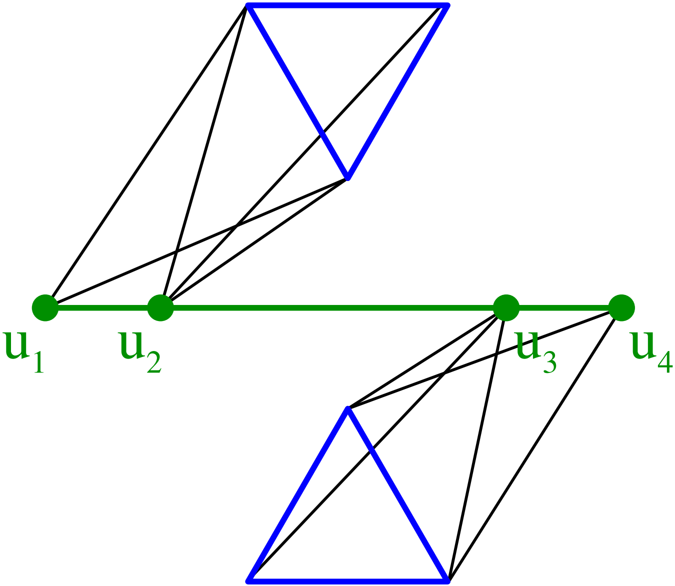


Claim 8

The red vertices together have at most $3 + 6 = 9$ neighbors, but $\sigma_2(G) \geq 4k - 2 = 10$.

Claim 8



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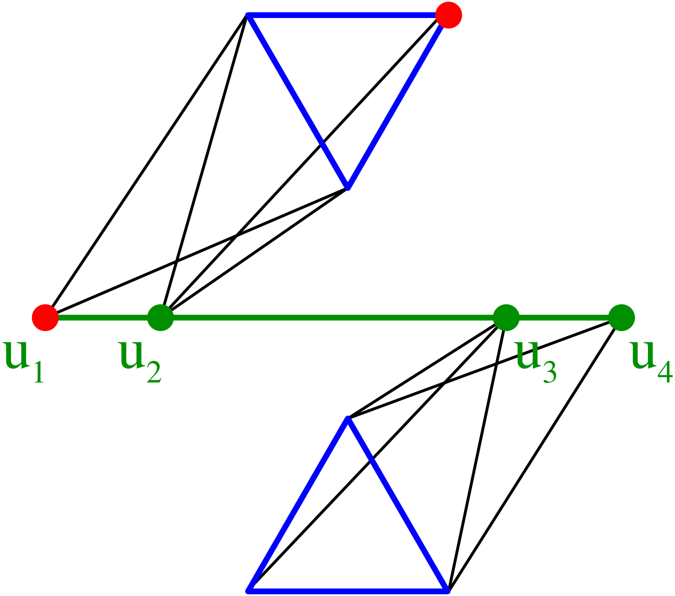

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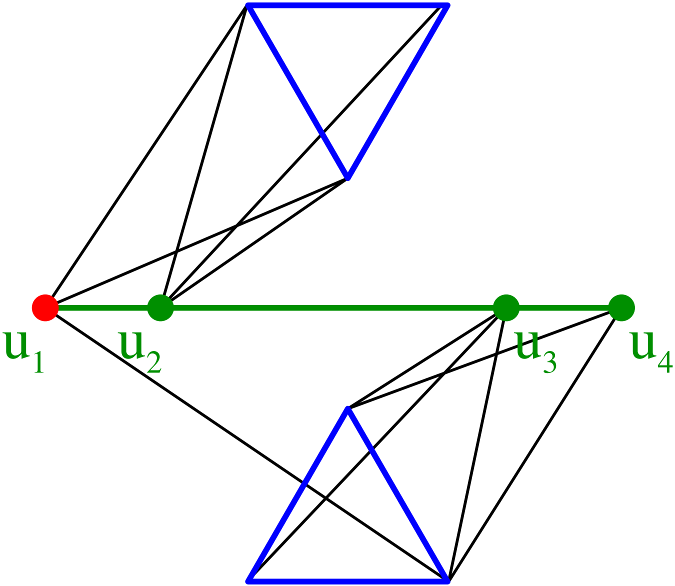

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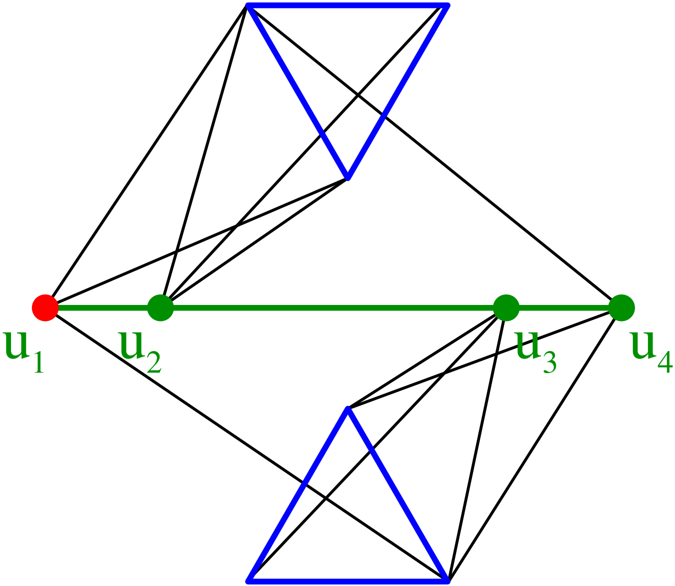

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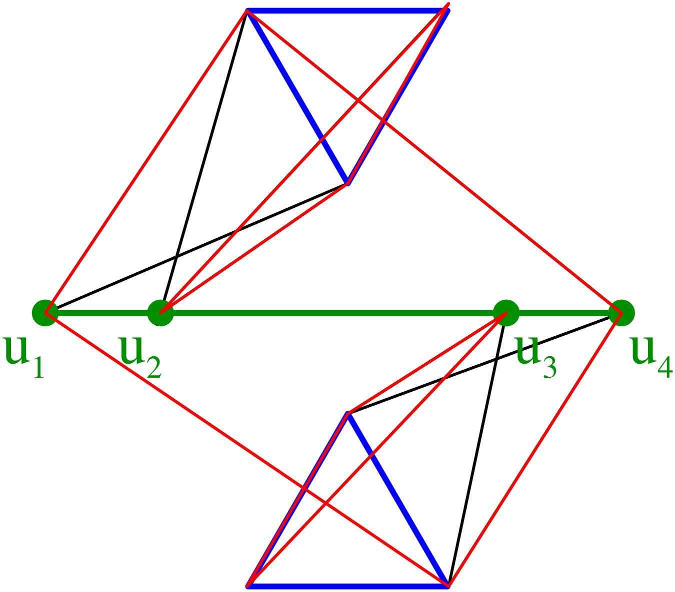

Goal 2: $|R| = 3$

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Claim 7

$k = 3$ and $\|C, F\| = 7$ for both $C \in \mathcal{C}$

Claim 8

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Proof of Therem 7

Goal (1)

$R := G - \mathcal{C}$ *is a path*

Goal (2)

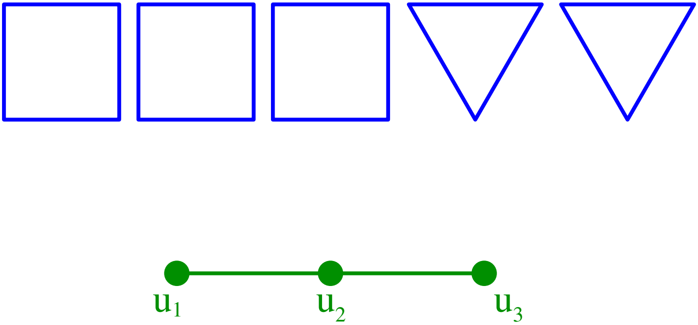
$|R| = 3$

Goal (3)

$|R| \geq 4$

Goal 3

We assume $|R| = 3$, and find a contradiction.



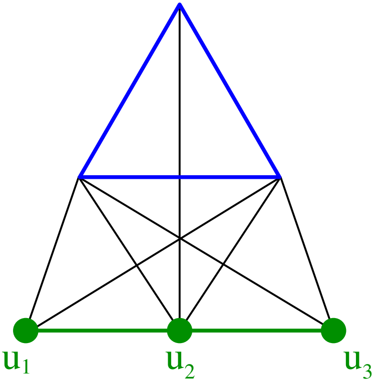
Claim 9

If C is a longest cycle in $\mathcal{C}$ and D is another cycle in $\mathcal{C}$, then $\|C, D\| \leq 2|C|$.

Claim 10

The longest cycle in $\mathcal{C}$ has four vertices.

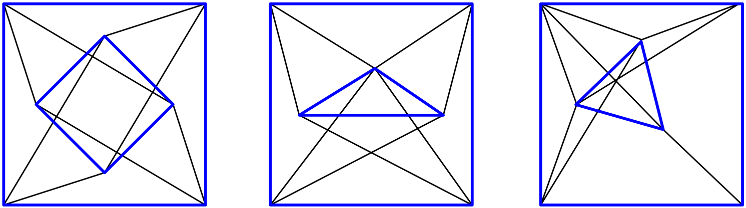
Goal 3



Claim 11

For any $D \in \mathcal{C}$, $\|D, R\| \leq 7$. If equality holds, $|D| = 3$ and $(R \cup D) \cong K_6 - K_3$.

Goal 3



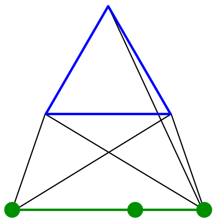
Claim 12

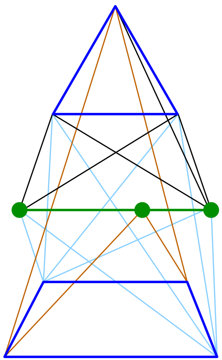
For all 4-cycles $C \in \mathcal{C}$, and all $D \in \mathcal{C} - C$, $||D, C|| \leq 8$.

Claim 13

For all 4-cycles $w_1 w_2 w_3 w_4 = C \in \mathcal{C}$ and all $d \in \mathcal{C} - C$, we have $2||\{w_1, w_3\}, D|| + ||\{w_2, w_4\}, D|| \leq 12$.

Goal 3





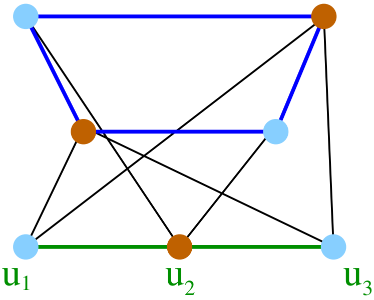
Claim 14

For every $D \in \mathcal{C}$, $||\{u_1, u_3\}, D|| \leq 4$.

Claim 15

For every $D \in \mathcal{C}$, $||\{u_1, u_3\}, D|| = 4$.

Goal 3



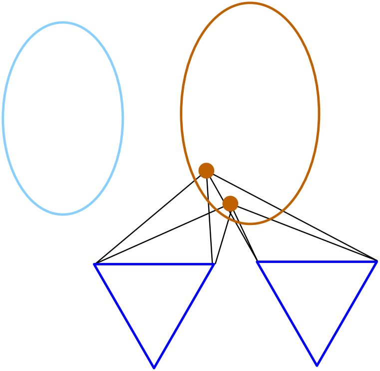
Claim 16

Given a 4-cycle $C \in \mathcal{C}$, $G[R \cup C] \cong K_{3,4}$

Claim 17

If $C_1, \ldots, C_s$ are the 4-cycles of $\mathcal{C}$, then $G[R \cup C_1 \cup \ldots \cup C_s] \cong K_{2s+1,2s+2}$. Call the smaller part A and the larger B.

Goal 3



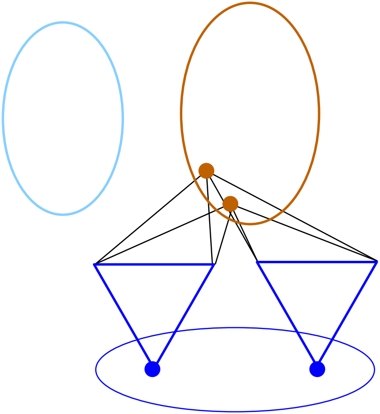
Claim 18

For every $b \in B$ and every 3-cycle $D \in \mathcal{C}$, $||b, D|| = 2$.

Claim 19

For every $b_1, b_2 \in B$ and every 3-cycle $D \in \mathcal{C}$,
$N(b_1) \cap D = N(b_2) \cap D$.

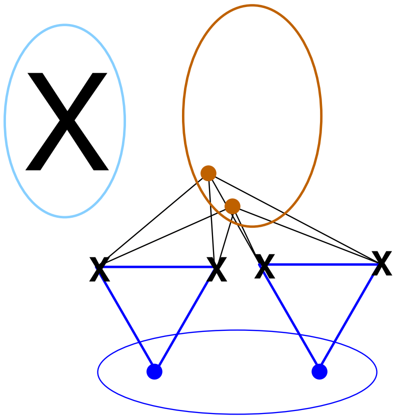
Goal 3



Claim 20

In the set of 3-cycles in $\mathcal{C}$, the vertices not adjacent to vertices from b are also not adjacent to each other.

Goal 3



Claim 21

G has an independent set of size
$|V(G)| - (2s+1) - 2(k-1-s) = |V(G)| - 2k + 1.$

Proof of Therem 7

Goal (1)

$R := G - \mathcal{C}$ *is a path*

Goal (2)

$|R| = 3$

Goal (3)

$|R| \geq 4$

Theorem (7)

[Kierstead, Kostochka, Y.]: Let $k \geq 3$, $n \geq 3k+1$, and let H be an n-vertex graph with $\sigma_2(H) \geq 4k-2$ and $\alpha(H) \leq n-2k$. Then H contains k vertex-disjoint cycles.

Thank you for listening!